Mathematical Physics II

By Dr. Issakha YOUM
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I. Mathematical Physics II

By Dr. Issakha Youm, Full Professor

II. Prerequisites

Before taking this module, the learner should have mastered the following notions: elements of vector calculus (vector sum, scalar multiplication of vectors, scalar product of two vectors), elements of analysis and series, partial derivatives, derivative of an implicit function, and total and exact differential equations.

III. Time

The duration of this module is 120 hours, allocated as follows:

- **Activity 1 (20 hours):** Vector algebra in $\mathbb{R}^3$
- **Activity 2 (30 hours):** Vector functions of one and two variables
- **Activity 3 (30 hours):** Physical fields
- **Activity 4 (40 hours):** Spatial integrals – applications in physics.

IV. Materials

In addition to the usual materials required for taking notes and making geometric drawings (e.g., notebook, pens, pencils, compass, square, protractor, ruler), the learner will need to have access to a computer with an Internet connection and a graphing calculator.

V. Module rationale

In physics, mathematical tools are used to formulate the laws of nature. Nevertheless, the fact is that teachers frequently lack the mathematical knowledge required to properly apply these tools in physics. In order to teach physics as a subject, it is important to take this Mathematical Physics course. It will provide you with the calculus rules that you need without having to take additional mathematics courses.
VI. Content

6.1 Overview

This Mathematical Physics II module builds on the Mathematical Physics I module. It addresses differential and integral calculus tools for functions (scalar and vector) of multiple variables. It reviews the areas of vectors, spatial geometry, vector functions, curves, surfaces, partial derivatives, multiple integrals and diverse applications such as surface and volume calculations. It also covers the notions of curvilinear integrals and surface integrals as well as the theorems of Gauss, Green and Stokes. It concludes with applications in wave theory and magneto-electric wave propagation. This last section, which explains some applications in the field of physics, gives the learner an idea of how mathematics is applied in practice.

6.2 Outline

The module is organized as follows:

- **Geometric vectors and vectors in \( \mathbb{R}^3 \):** Cartesian coordinate system for a three-dimensional space, vector position of a point; scalar, vector and triple (or mixed) products: properties and geometric and physical interpretations.

- **Vector functions of one and two variables:** derivatives and derivation rules, defined integral and vector integral; parametric curves, tangent vector, arc length, curvature, torsion and the Serret-Frenet system; parametric surfaces, tangent plane to a surface; cinematic application: velocity and acceleration vectors.

- **Physical fields:** scalar and vector fields, level curve and level surface, vector field, gradient field, and differential operators: divergence, rotational and Laplacian.

- **Spatial integrals:** curvilinear integrals, surface and volume integrals; theorems of Gauss, Green and Stokes.

- **Applications in physics:** wave equations, electromagnetic wave propagation in a vacuum and material media.
Johann Peter Gustav Lejeune Dirichlet (1805–1859)

6.3 Graphic Organizer

- Algebra of vectors in $\mathbb{R}^3$
- Vector functions
- Physical fields
- Spatial integrals
- Applications in physics
VII. General Objectives

- Understand basic notions of differential calculus and vector calculus.
- Understand tools of differential and integral calculation applied to functions (scalar and vector) of multiple variables
- Use vector derivatives
- Use the gradient and other vector analysis operators, with a focus on geometric and physical interpretations.

VIII. Specific Learning Objectives

<table>
<thead>
<tr>
<th>Unit</th>
<th>Learning objectives</th>
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| 1. Vector algebra in $\mathbb{R}^2$ | Learners should be able to:  
Define the modulus, support and standard of a vector.  
Calculate the sum of two vectors.  
Understand the relationships between a vector space and a physical space and its vectors.  
Define a vector position and a radius vector.  
Determine the components of a vector and the coordinates of a point.  
Determine the unit vector in a given direction.  
Calculate a scalar product, a vector product and a mixed product.  
Provide the formula for a double vector product.  
Geometrically interpret a scalar product, a vector product and a triple vector (mixed) product. |
| 2. Vector functions | • Define vector value functions (of one or two variables) and understand the notions of vector limits and vector continuity.  
• Calculate the derivatives and integrals of vector functions.  
• Apply the derivation rules for vector functions, particularly the derivation of single vector products.  
• Geometrically interpret the notions of the derivation and integration of vector functions.  
• Geometrically interpret vector functions for parametric curves and surfaces.  
• Define a normal vector to a surface.  
• Define curvature and torsion of a spatial curve.  
• Calculate the velocity and acceleration vectors for a trajectory in a Frenet-frame (moving frame). |
| 3. Physical fields and their derivatives | • Define a scalar field and a vector field.  
• Define a level curve, a level surface and field lines.  
• Determine the equations that define these notions and solve simple cases.  
• Define the gradient of a scalar field.  
• Calculate the gradient of a scalar function.  
• Define the divergence and rotational of a vector field.  
• Calculate the divergence and rotational of a vector field.  
• Define the nabla operator (or del).  
• Determine the conditions for a vector field to be a gradient field. |
| 4. Spatial integration | • Calculate a curvilinear integral by curve parameterization.  
• Calculate the circulation of a vector field.  
• Define the integral condition for a vector field to be a gradient field and its operation.  
• Determine the derivation function for a gradient field using the circulation method.  
• Calculate a double integral for any parameterized surface.  
• Calculate a vector field flux  
• Calculate volume integrals.  
• Geometrically interpret the nabla operator. |
| 5. Applications in physics | • Explain the notion of an electromagnetic field.  
• Apply mathematical theories of integrals.  
• Apply the local laws of electromagnetism.  
• Describe wave propagation in media.  
• Describe the phenomena of reflection, refraction, dispersion and absorption. |
IX. Pre assessment

9.1 Predictive assessment

Title of the predictive assessment: Test on the prerequisites for the Mathematical Physics II module

Purpose: This test assesses whether the learner has the required knowledge to understand the Mathematical Physics II module.

QUESTIONS

1. In a two-dimensional space, a straight line is represented by the equation $3x - 2y + 6 = 0$. A direction vector for this straight line is the vector with the components:
   A - (3,2)    B - (2,3)    C - (3,-2)    D - (-2,-3)

2. In a three-dimensional space, we have the Cartesian base vectors $\vec{a} (3,-1,2)$; $\vec{b} (-1,3,3)$; $\vec{c} (5,4,-1)$. The coordinates of $\vec{A} = 3\vec{a} - 2\vec{b} - \vec{c}$ are:
   A - (3,-2-1)   B - (3,-1,2) C - (6,-13,1) D - (5,3,-2)

3. The vector sum of $\vec{B}C - 2\vec{BA} + \vec{C}D - \vec{AD}$ is equal to:
   A - \overrightarrow{AB}    B - \overrightarrow{BD}    C - 0    D - \overrightarrow{CD}

4. In a Cartesian system, the vectors $\vec{a} (-1,-2,1)$ and $\vec{b} (1,2,5)$ are:
   A - collinear    B - opposed    C - orthogonal    D - none of these

5. In a Cartesian system, the vectors $\vec{a} (-3,-1,4)$ and $\vec{b} \left(\frac{1}{2}, \frac{1}{6}, -\frac{2}{3}\right)$ are:
   A - collinear    B - opposed    C - orthogonal    D - none of these
6. Consider a parallelogram ABCD. Which of the following obtains exact equality?

A - \( \overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB} \)
B - \( \overrightarrow{DA} + \overrightarrow{DC} = \overrightarrow{DB} \)
C - \( \overrightarrow{AD} + \overrightarrow{AC} = \overrightarrow{AB} \)
D - \( \overrightarrow{DB} + \overrightarrow{DC} = \overrightarrow{BC} \)

7. If O represents the centre of a parallelogram ABCD, which vector is equal to the vector \( \overrightarrow{OD} - \overrightarrow{BA} \)?

A - \( \overrightarrow{OA} \)
B - \( \overrightarrow{0} \)
C - \( \overrightarrow{OB} \)
D - \( \overrightarrow{OC} \)

In questions 8 and 9, consider the plane (P) of the equation \( 2x - 3y + z = 2 \).

8. Which of these points is located in the plane (P)?

A - (0,0,2)  B - (0,0,0)  C - (0,-1,1)  D - (1,1,1)

9. Which of these vectors is a unit vector perpendicular to the plane (P)?

A - \( \left(-\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right) \)
B - (0,0,1)
C - \( \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \)
D - \( \left(-\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) \)

10. For the function \( f(x) = \sqrt{2x - 5} \), which of the following is the complete definition of this function?

A - \( \mathbb{R} \)
B - \( \mathbb{R} - \left\{-\frac{5}{2}; \frac{5}{2}\right\} \)
C - \( \left[\frac{5}{2}; +\infty\right[ \)
D - \( \left[-\infty; -\frac{5}{2}\right[ \cup \left[\frac{5}{2}; +\infty\right[ \)
11. For the function \( f(x) = \frac{2}{\sqrt{x^3 + 2}} \), which of the following is the first derivative of this function?

A. \( \frac{3x}{\sqrt{x^3 + 2}} \)

B. \( -\frac{3x^2}{\sqrt{x^3 + 2}} \)

C. \( -\frac{3x^2}{\sqrt{x^3 + 2}} \)

D. \( -\frac{3x^2}{(x^3 + 2)^{3/2}} \)

12. For the function \( f(x) = \frac{1}{1 - x} \),

a) the series development around \( x = 0 \) is which of the following?

A. \( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \)

B. \( 1 + x + x^2 + x^3 + x^4 + \ldots \)

C. \( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \)

D. \( 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots \)

b) The convergence field for this series is which of the following?

A. \( ]-1,1[ \)

B. \( \mathbb{R} \)

C. \( ]1,\infty[ \)

D. \( \mathbb{R} \{1\} \)
13. A primitive of the function \( f(x) = 2\sin 2x - 3\cos x \) is which of the following?

A. \( \frac{1}{2}\cos x - \frac{1}{3}\sin x \)
B. \( \frac{1}{2}\cos x - 3\sin x \)
C. \( \frac{1}{2}\cos 2x - 3\sin x \)
D. \( -\cos 2x - 3\sin x \)

14. The integral \( I = \int_0^R 4\pi x\sqrt{R^2 - x^2} \, dx \) is equal to which of the following?

A. 0
B. \( \pi R^3 \)
C. \( \frac{4\pi R^3}{3} \)
D. \( 4\pi R^2 \)

15. Of the following two-variable functions with respect to the variables \( x \) and \( y \), indicate which ones are first and second order partial derivatives.

a) \( xy \)
A - 0
B - \( x \)
C - \( y \)
D - \( xy \)
E - 1
F - \( x + y \)
G - \( 1 + x \)
H - \( 1 + y \)

b) \( 2x^2y + y^3 \)
A - \( 2x^2 + y^2 \)
B - \( 4y \)
C - \( 2xy + 3y^2 \)
D - \( 4x \)
E - \( 4y \)
F - \( 6y \)
G - \( 4xy \)
H - \( 2x^2 + 3y^2 \)
c) \((x^2 + y^2)^{1/2}\)

A - \(x(x^2 + y^2)^{-1/2}\)

B - \(y^2(x^2 + y^2)^{-3/2}\)

C - \(x^2(x^2 + y^2)^{-3/2}\)

D - \(y(x^2 + y^2)^{-1/2}\)

E - \(xy(x^2 + y^2)^{1/2}\)

F - \(xy(x^2 + y^2)^{-1/2}\)

G - \(x^2y^2(x^2 + y^2)^{1/2}\)

H - \(-xy(x^2 + y^2)^{-3/2}\)

16. The differential of the function \(f(x, y, z) = x^2 - 2xy - z^2\) is given by:

A - \(2(x - y)dx - 2xdy - 2zdz\)

B - \(2xdx - 2xdy - 2zdz\)

C - \(2xdx - 2ydy - 2zdz\)

D - \(2(x + y)dx - 2xdy - 2zdz\)

17. Of the following differential forms, indicate which ones are exact differentials.

A. \(\sin y \, dx + \cos x \, dy\)

B. \(-ydx\)

C. \(xdy - 3ydx\)

D. \(\frac{ydx - xdy}{y^2}\)
African Virtual University

18. In a curve equation, \( f(x, y) = x^3 - 2xy + 2y^2 - 1 = 0 \). The equation of the tangent \( \Gamma \) at point \( M(1,1) \) is given by which of the following expressions?

A. \( 2x + 1 \)

B. \( -\frac{1}{2}x + \frac{3}{2} \)

C. \( \frac{1}{2}x - \frac{3}{2} \)

D. \( -2x + \frac{1}{2} \)

19. A surface \( (S) \) has the equation \( z = x^2 - 3y + 2 \). If \( z = 2 \), the level surface contains which of the following points?

A - (1,1,2)    B - (1,2,-4)    C - (2,1,2)    D - (3,3,2)

20. A surface \( (S) \) has equation \( z = \frac{x^2}{y} \). The equation of the plane tangent to the surface \( (S) \) at point \( M(1,1,1) \) is which of the following?

A. \( -2X + Y + Z = 0 \)

B. \( X + Y + Z - 1 = 0 \)

C. \( X + Y + Z = 0 \)

D. \( 2X + 2Y + Z - 3 = 0 \)
African Virtual University  Answer Key

1. The direction vector of the straight-line equation \( ax + by + d = 0 \) is 
\[ \mathbf{u} = -b \mathbf{i} + a \mathbf{j}, \] 
which is the case for the vector with components (2,3).

The answer is B.

2. The vector 
\[ \mathbf{A} = 3\mathbf{a} \mathbf{b} \mathbf{c} = 6\mathbf{i} 13 \mathbf{j} + \mathbf{k} . \]

The answer is C.

3. We have 
\[ \mathbf{B} \mathbf{C} - 2 \mathbf{B} \mathbf{A} + \mathbf{C} \mathbf{D} - \mathbf{A} \mathbf{D} = \mathbf{B} \mathbf{C} + \mathbf{C} \mathbf{D} + \mathbf{D} \mathbf{A} - 2 \mathbf{B} \mathbf{A} = \mathbf{B} \mathbf{A} - 2 \mathbf{B} \mathbf{A} = -\mathbf{B} \mathbf{A} = \mathbf{A} \mathbf{B} . \]

The answer is therefore A.

4. Two non-null vectors are orthogonal if, and only if, their scalar product equals zero: 
\[ \mathbf{a} \cdot \mathbf{b} = -1 - 4 + 5 = 0 . \]

The answer is C.

5. The answer is A. Because we have 
\[ \mathbf{a} = -6 \mathbf{b} . \]

6. The answer is B. As illustrated in the figure below, we have 
\[ \mathbf{D} \mathbf{A} + \mathbf{D} \mathbf{C} = \mathbf{a} - \mathbf{b} = \mathbf{D} \mathbf{B} . \]

7. The answer is D. As illustrated in the figure above, we have 
\[ \mathbf{O} \mathbf{D} - \mathbf{B} \mathbf{A} = \mathbf{B} \mathbf{O} + \mathbf{A} \mathbf{B} = \mathbf{A} \mathbf{O} = \mathbf{O} \mathbf{C} . \]

8. The point in the plane must satisfy the equation of the plane. Here, it is the point with coordinates is (0,0,2), so the answer is A.

9. The plane of the equation \( ax + by + cz + d = 0 \) can have the normal vector 
\[ \mathbf{N} = a \mathbf{i} + b \mathbf{j} + c \mathbf{k} , \]
and the unit vector 
\[ \mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{2}{\sqrt{14}} \mathbf{i} - \frac{3}{\sqrt{14}} \mathbf{j} + \frac{1}{\sqrt{14}} \mathbf{k} . \]

Therefore the answer is D.
10. This function can only be defined if \( 2x - 5 \geq 0 \Rightarrow x \geq \frac{5}{2} \). The complete definition of the function is therefore \( D_f = \left[ \frac{5}{2}, \infty \right] \).

The answer is C.

11. The derivative of the function is readily calculated by assuming that \( u = x^3 + 2 \) which gives \( \frac{du}{dx} = 3x^2 \) and \( f'(x) = f'(u)u'(x) = -3x^2(x^3 + 2)^{-3/2} \).

The answer is D.

12. a) The development of a series for the function \( f(x) \) around \( x = 0 \) is written as:

\[
\begin{align*}
  f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \\
  \text{in this case with:} \\
  f(0) &= 1; \quad f'(0) = 1; \quad f''(0) = 2!; \quad f'''(0) = 3!; \quad \ldots
\end{align*}
\]

whence:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots
\]

The answer is therefore B.

b) The convergence field for a series is determined by:

\[
\lim_{n \to \infty} \left| \frac{C_n}{C_{n+1}} \right| = R , \text{ where } C_n \text{ is the coefficient of the } n^{th} \text{ term in the series.}
\]

If:

- \( R = \infty \), then the series converges for all values of \( x \);

- \( 0 < R < \infty \), then the series converges for \( |x| < R \);

- \( R = 0 \), then the series converges only for \( x = 0 \).
In this example, we have \( R = 1 \), so the series converges for \( |x| < 1 \), and the answer is A.

13. We have

\[
\int (2 \sin 2x - 3 \cos x) \, dx = 2 \int \sin 2x \, dx - 3 \int \cos x \, dx = -\cos 2x - 3 \sin x + C
\]

The answer is D. Remember that a primitive of \( a \sin bx \) is the function \(-\frac{a}{b} \cos bx\) and a primitive of \( a \cos bx \) is the function \( \frac{a}{b} \sin bx \).

14. If we change the variable \( X = R^2 - x^2 \), which gives \( dX = -2x \, dx \), this gives the new integration boundaries \( X(0) = R^2 \) and \( X(R) = 0 \). Hence, the integral

\[
I = -2\pi \int_0^R \sqrt{X} \, dX = 2\pi \int_0^{R^2} \sqrt{X} \, dX = \frac{4\pi R^3}{3}.
\]

The answer is C.

15. The first partial derivatives are \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \), and the second partial derivatives are given by

\[
\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.
\]

Hence:

a) \( \frac{\partial f}{\partial x} = y \), \( \frac{\partial f}{\partial y} = x \), then \( \frac{\partial^2 f}{\partial x^2} = 0 \), \( \frac{\partial^2 f}{\partial y^2} = 0 \), and finally

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1.
\]

The answers are therefore cases C, B, A and E, respectively.
b) \( \frac{\partial f}{\partial x} = 4xy \), \( \frac{\partial f}{\partial y} = 2x^2 + 3y^2 \), then \( \frac{\partial^2 f}{\partial x^2} = 4y \), \( \frac{\partial^2 f}{\partial y^2} = 6y \), and finally \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 4x \). 

The answers are therefore cases G, H, E, F and D, respectively.

c) \( \frac{\partial f}{\partial x} = x(x^2 + y^2)^{1/2} \), \( \frac{\partial f}{\partial y} = y(x^2 + y^2)^{1/2} \), then \( \frac{\partial^2 f}{\partial x^2} = y^2(x^2 + y^2)^{-3/2} \), \( \frac{\partial^2 f}{\partial y^2} = x^2(x^2 + y^2)^{-3/2} \), and finally \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -xy(x^2 + y^2)^{-3/2} \).

The answers are therefore cases A, D, B, C and H.

16. The differential of the function \( f(x, y, z) \) is given by:

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz
\]

Thus, by replacing the partial derivatives by their expressions, we obtain:

\[
df = (2x - 2y) \, dx - 2x \, dy - 2z \, dz
\]

The answer is A.

17. A differential form \( dZ = F(x, y) \, dx + G(x, y) \, dy \) is exact if there is equality:

\[
\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}
\]

Only the differential form given in case D satisfies this condition. In effect, we have \( F(x, y) = \frac{1}{y} \) et \( G(x, y) = -\frac{x}{y^2} \) d'où \( \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = -\frac{1}{y^2} \).
18. The point M (1,1) is indeed located on the curve $\Gamma$, since we have: $f(1,1) = 0$.

Moreover, we have $\frac{\partial f}{\partial y} = 4y - 2x = 0$ at point A (1,1). The function $f(x, y)$ therefore implicitly defines $y$ as a function of $x$ at this point: $y = \varphi(x) \iff f(x, y) = 0$ with $\varphi(1) = 1$.

Hence $\varphi'(x) = -\frac{\partial f}{\partial x}(1,1) = -\frac{1}{2}$ and the tangent to the curve at point A (1,1) is represented by the equation: $y - 1 = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x + \frac{3}{2}$.

The answer is therefore B.

19. The answer is D. We have: $3^2 - 3.3 + 2 = 2$.

20. The two-variable function $z = \varphi(x, y) = \frac{x^2}{y}$ defining the surface (S) is an implicit function given by the equation: $f(x, y, z) = yz - x^2 = 0$. The equation of the plane tangent to the surface (S) of equation $f(x, y, z) = 0$ at a point $M(x, y, z)$ is given by:

$$(Z - z) \frac{\partial f}{\partial z} = -(X - x) \frac{\partial f}{\partial x} - (Y - y) \frac{\partial f}{\partial y}$$

or

$$(X - x) \frac{\partial f}{\partial x} + (Y - y) \frac{\partial f}{\partial y} + (Z - z) \frac{\partial f}{\partial z} = 0$$

In this case, $-2x(X - x) + z(Y - y) + y(Z - z) = 0$.

Thus, by replacing the coordinates of the point by their values, we obtain:

$-2(X - 1) + (Y - 1) + (Z - 1) = 0$
and after development: \(-2X + Y + Z = 0\). The answer is therefore A.

**Guidelines for learners (100–200 words)**

If you scored at least 75%, you appear to be interested in this module, and I encourage you to continue.

If you scored between 50% and 75%, your results are very encouraging. Keep up the good work.

If you scored between 35% and 50%, this is far from a perfect score. However, you have demonstrated your will to succeed. This kind of enthusiasm is invaluable. Nevertheless, the fact is, although you have chosen a fascinating field, it requires a lot of work. You have some catching up to do, but if you are persistent, you will succeed in the end.

If you scored less than 35%, you have a great deal of work ahead of you. In addition to taking this module, you should review your previous courses.
X. Learning activities

Learning activity 1

Title of the activity: Vector algebra in $\mathbb{R}^3$

Learning time

20 hours

Guidelines: If you score at least 75% on this activity, you have done a great job, and you can continue.

If you score less than 50%, you should review the suggested readings and re-do the activity.

If you score between 50% and 75%, you have worked hard, but you must re-double your efforts in future.

Specific objectives

Upon completion of this activity, the learner should be able to:

• Define the modulus of a vector.
• Calculate the sum of two vectors.
• Understand the relationships between a vector space and a physical space and its vectors.
• Define a vector position and a radius vector.
• Determine the components of a vector and the coordinates of a point.
• Determine the unit vector in a given direction.
• Calculate a scalar product, a vector product and a mixed product.
• Provide the formula for a double vector product.
• Geometrically interpret a scalar product, a vector product and a triple vector (mixed) product.

Summary of the activity

This activity addresses vectors. A number of physical phenomena can be characterized in terms of vector quantities. For example, to move an object, you must know not only how much force is required (i.e., intensity of the force), but also the direction (i.e., sense of the force). Only mathematical terms called vectors can be used to describe these physical quantities. The purpose of this activity is to familiarize you with vectors and how to use them.
Useful resources


Useful links

http://en.wikipedia.org/wiki/Vector
http://en.wikipedia.org/wiki/Vector_space
http://www-math.mit.edu/~djk/18_022/chapter02/section02.html

Description of the activity

This activity must be completed in a series of steps designed to help the learner grasp the different themes: vector addition, scalar multiplication, scalar product, vector product, mixed vector product and double vector product. There are five exercises to complete.
Formative assessment

Learners must work in collaboration to complete all the exercises.

Exercises 1 and 2 count for 15% each of the total mark, exercise 3 counts for 20%, and exercises 4 and 5 count for 25% each.

Exercises

Exercise 1

In a space represented by the orthonormal system \((O; \vec{i}, \vec{j}, \vec{k})\), a particle travels from point A with coordinates \((2, 3, 5)\) to point B with coordinates \((3, 8, 6)\). In questions 1.a) and 1.b) below, check the sets of values that represent

a) the components of the vector \(\overrightarrow{OA}\)
   
   A - (1, 3, 5)
   B - (2, 3, 5)
   C - (3, 8, 6)

b) the components of the displacement vector \(\overrightarrow{AB}\)
   
   A - (3, 8, 6)
   B - (2, 3, 5)
   C - (1, 5, 1)

c) Which of the following represents the displacement vector for the particle \(\overrightarrow{AB}\) ?
   
   A - \(\overrightarrow{OA} + \overrightarrow{OB}\)
   B - \(\overrightarrow{OA} - \overrightarrow{OB}\)
   C - \(\overrightarrow{OB} - \overrightarrow{OA}\)
Exercise 2

In a space represented by the orthonormal system \((O; \mathbf{i}, \mathbf{j}, \mathbf{k})\), the vectors are defined by the following equations:

\[
\vec{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}; \quad \vec{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}; \quad \vec{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}; \quad \vec{p} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}; \quad \vec{q} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}; \quad \vec{r} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}; \quad \vec{s} = 5\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}; \quad \vec{t} = 4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}
\]

Which of the following vectors are parallel?

A- \( \vec{v} \) and \( \vec{u} \)
B- All the vectors are parallel
C- \( \vec{w} \) and \( \vec{p} \)
D- \( \vec{p} \) and \( \vec{s} \)
E- \( \vec{v} \) and \( \vec{r} \)
F- \( \vec{v} \) and \( \vec{w} \)
G- \( \vec{w} \) and \( \vec{r} \)
H- \( \vec{t} \) and \( \vec{q} \)
I- \( \vec{w} \) and \( \vec{q} \)
J- \( \vec{q} \) and \( \vec{u} \)

Exercise 3

In a space represented by the orthonormal system \((O; \mathbf{i}, \mathbf{j}, \mathbf{k})\), the vectors are defined by the following equations:

\[
\vec{a} = 3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}; \quad \vec{b} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}; \quad \vec{c} = 2\mathbf{j} - \mathbf{k}
\]

Indicate which answer(s) correspond to

a) \( \vec{a} \cdot \vec{b} \)

b) \( |\vec{a}| \)

c) \( |\vec{b}| \)

d) \( |\vec{a} \times \vec{b}| \)
Which of the following is the angle formed by the vectors $\vec{a}$ and $\vec{c}$?

A. 60°
B. 141.2°
C. 50.3°
D. 151.3°

Which of the following are the components of the unit vector $\vec{u}$ for the vector $\vec{a} + \vec{c}$?

A. (0.68, 0.27, -0.68)
B. (0.35, 0.46, -0.81)
C. (0.71, -0.71, 0)
D. (1,1,1)

Which of the following is (or are) perpendicular to the vector $\vec{d} = -3 \vec{j} - \vec{k}$?

A. $\vec{a}$
B. $\vec{b}$
C. $\vec{c}$
D. $\vec{c}$
Exercise 4
Which of the following equations are mathematically incorrect? Justify your answer.

A- \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)

B- \( ||\mathbf{u} + \mathbf{v}|| = ||\mathbf{v} + \mathbf{u}|| \)

C- \( ||\mathbf{u}|| + \mathbf{v} = ||\mathbf{u} + \mathbf{v}|| \)

D- \( ||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}|| \)

E - \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \)

F - \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \times (\mathbf{u} \cdot \mathbf{w}) \)

G - \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \)

H - \( (\mathbf{u} - \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{w} \times \mathbf{v} \)

Exercise 5
For the vector \( \mathbf{w} = (\mathbf{u} \times \mathbf{v}) \times \mathbf{u} \), are the following statements true or false?

1- This vector is located in the plane formed by the vectors \( \mathbf{u} \) and \( \mathbf{v} \).
   A- TRUE
   B- FALSE

2- This vector is equal to the vector \( ||\mathbf{u}||^2 \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \).
   A- TRUE
   B- FALSE
Learning activities

- The learner must duly complete the following steps:

**Step 1**: Objectives to achieve (time: 10 minutes)
  - First, carefully read the objectives to achieve.
  - Write these objectives in your notebook and highlight or underline the key points that you must focus on.

**Step 2**: Obtain the required background knowledge by doing the required reading and visiting the Internet sites. This individual learning enables you to acquire the knowledge you need to achieve the objectives (time: 10 hours, using good time management).
  - Carefully read the chapter that covers these objectives (required reading).
  - In your notebook, write the key points of the course that you must focus on.
  - Consult the suggested references and visit the sites listed below in the Useful Links section.

**Step 3**: Understanding the course (group learning) (time: 5 hours)
  - Under your tutor’s guidance, share information using chat to make sure that you understand the material.
  - Work in collaboration with a group organized by your tutor.
  - Comply with the order and resolution time of the exercises, as indicated by your tutor.

**Step 4**: Assessment (time: 5 hours)
  - Each group will select a reporter, who will send the course instructor an email with an attached file containing the group’s report. This report will include the names of all the group members and the solutions to each exercise. The reporter may change from one exercise to another.
Answer key

Solution to exercise 1

a) The answer is case B. The vector $\overrightarrow{OA}$ corresponds to point A in the reference frame, and its components are identical to the coordinates of point A.

To learn more and reinforce what you have learned, review the definition of the components of a vector in an orthonormal reference frame (or system).

b) As it travels from A to B, the particle is displaced 1 unit in the positive $x$ direction, 5 units in the positive $y$ direction, and finally 1 unit in the positive $x$ direction. Thus, the vector $\overrightarrow{AB} = \hat{i} + 5 \hat{j} + \hat{k}$. Therefore, only answer C is correct.

To learn more and reinforce what you have learned, show that the vector $\overrightarrow{MM'}$ representing the displacement of a point $M(x,y,z)$ to point $M'(x',y',z')$ is given by the equation: $\overrightarrow{MM'} = (x'-x) \hat{i} + (y'-y) \hat{j} + (z'-z) \hat{k}$.

c) The corresponding vector is $\overrightarrow{OB} - \overrightarrow{OA}$. Therefore, case C is correct. In fact, using Charles’ equation, we can write: $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$, where vector $\overrightarrow{AO}$ is the inverse of vector $\overrightarrow{OA}$. Thus, we can write: $\overrightarrow{AO} = - \overrightarrow{OA}$, where $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

To learn more and reinforce what you have learned, perform Charles’ equation and review the section on vector addition using the parallelogram method. Graphically represent a vector and its opposite.

Solution to exercise 2

The $\lambda \in \mathbb{R}$ conditions must be such that, between any two vectors $\vec{a}$ and $\vec{b}$, the relationship is $\vec{a} = \lambda \vec{b}$, i.e., if $\vec{a}$ and $\vec{b}$ are parallel. Thus, we obtain $\vec{u} = 3 \vec{v}$; $\vec{r} = -2 \vec{w}$; $\vec{s} = -5 \vec{p}$; $\vec{t} = 4 \vec{q}$. The correct answers are therefore cases A, D, G and H.
Solution to exercise 3

1- By definition, the scalar product of two vectors gives a scalar quantity. Here, the vectors are expressed in an orthonormal system, so we can use the analytical expression of the scalar product: \[
\mathbf{u} \cdot \mathbf{v} = xx' + yy' + zz'.
\] We then obtain:
\[
\mathbf{a} \cdot \mathbf{b} = -8 ; \quad |\mathbf{a}| = 7 ; \quad |\mathbf{b}| = 4.58 ; \quad |\mathbf{a} + \mathbf{b}| = 7.35
\]
The corresponding cases are respectively C, D, E and F.

2) Using the equality of the analytic and geometric expressions of the scalar product, we can calculate the angle between the two vectors, or \(\theta\). We then have:
\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}||\mathbf{c}|}.
\]
If \(\cos \theta = 0.64\), then \(\theta = 50.3^\circ\). The correct answer is therefore case C.

3) The unit vector \(\mathbf{u}\) of the vector \((\mathbf{a} + \mathbf{c})\) is such that:
\[
\mathbf{u} = \frac{\mathbf{a} + \mathbf{c}}{|\mathbf{a} + \mathbf{c}|}
\]
Given \(\mathbf{u} (0.35, 0.46, -0.81)\), the answer is case B. Nevertheless, we must verify that the modulus of this vector is equal to 1: \(\sqrt{(0.35)^2 + (0.46)^2 + (-0.81)^2} \approx 1\), as expected.

4) The condition for orthonormality between two non-null vectors is that the scalar product of these two vectors is null, in this case:
\[
\mathbf{a} \cdot \mathbf{d} = 0 ; \quad \mathbf{b} \cdot \mathbf{d} = 6 \quad \text{et} \quad \mathbf{c} \cdot \mathbf{d} = 5.
\]
The correct answer is A.
Solution to exercise 4:

A- The equation is correct: the sum of the two vectors is a commutative operation.

B- Given the previous equality and the equality of the moduli of the two equal vectors, this equation is also true.

C- This equation is incorrect, because we cannot sum a vector and a scalar quantity (the modulus of a vector).

D- This equation is true only if the vectors are collinear.

E- These two vectors are not equal since \( \mathbf{u} \land (\mathbf{v} \land \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \)

\( \mathbf{u} \land (\mathbf{v} \land \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \)

ant and d

\( (\mathbf{u} \land \mathbf{v}) \land \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{v} \cdot \mathbf{w}) (\mathbf{u} \land \mathbf{v}) \land \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{v} \cdot \mathbf{w}) \mathbf{u} \)

yet \( (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \) \( (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \) need not be equal to \( (\mathbf{v} \cdot \mathbf{w}) \mathbf{u} (\mathbf{v} \cdot \mathbf{w}) \mathbf{u} \) for any arbitrary set \( \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \). The statement is hence not true in general. It is only true in special cases. The vector product is not associative.

F- This equation is incorrect. The right side does not make sense, because a vector product can apply only to vectors and not to scalars.

G- The statement is correct. The mixed product is geometrically interpreted as the volume of a parallelepiped constructed on the three vectors:

- \( \mathbf{v} \land \mathbf{w} = (\text{area of the parallelogram constructed on } \mathbf{v} \text{ and } \mathbf{w}) \mathbf{n} \); where \( \mathbf{n} \)

is the unit vector perpendicular to \( \mathbf{v} \) and \( \mathbf{w} \).

- Hence, \( \mathbf{u} \cdot \mathbf{n} \) gives the height of the parallelepiped. Thus, the volume of the parallelepiped is given by the surface of the base multiplied by the height.

H- The equation is correct, because the vector product is distributive with respect to the vector addition, and moreover, it is anticommutative.
Solution to exercise 5

1- The vector \( \overrightarrow{w} \) is specifically perpendicular to the vector \( \overrightarrow{u} \times \overrightarrow{v} \), and, like all vectors perpendicular to the vector \( \overrightarrow{u} \times \overrightarrow{v} \), it is located in the plane formed by the vectors \( \overrightarrow{u} \) and \( \overrightarrow{v} \). The statement is true.

2- The two vectors are equal, so, by applying the previously established equation
\[
(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c} = (\overrightarrow{c} \cdot \overrightarrow{a})\overrightarrow{b} - (\overrightarrow{c} \cdot \overrightarrow{b})\overrightarrow{a},
\]
we obtain:
\[
\overrightarrow{w} = ||\overrightarrow{u}||^2 \overrightarrow{v} - (\overrightarrow{u} \cdot \overrightarrow{v})\overrightarrow{u}.
\]
The statement is therefore true.

Self assessment

Learners should make a note of their difficulties and errors when seeking the solutions to these exercises in order to avoid repeating them. They should review the areas of the course that they have not completely understood and prepare a summary assessment.

Instructor’s guide

The instructor corrects the group reports. He or she enters the corrections to the exercise answers in a work space that the learners can access. Corrections should be accompanied by adequate feedback on the errors made in the reports. The mark for the group is assigned to all group members, and counts for 20% of the final mark for the module.
Learning activity 2

Title of the activity: Vector functions

Learning time

30 hours

Guidelines: If you score at least 75% on this activity, you have done a good job, and you should continue.

If you score less than 50%, you should review the suggested readings and re-do the activity.

If you score between 50% and 75%, you have worked hard, but you must re-double your efforts in future.

Specific objectives

At the end of this activity, you should be able to:

• Define vector value functions (of one or two variables) and understand the notions of vector limits and vector continuity.
• Calculate the derivatives and integrals of vector functions.
• Apply the derivation rules for vector functions, particularly the derivation of single vector products.
• Geometrically interpret the notions of the derivation and integration of vector functions.
• Geometrically interpret vector functions for parametric curves and surfaces.
• Define a normal vector to a surface.
• Define curvature and torsion of a spatial curve.
• Calculate the velocity and acceleration vectors for a trajectory in a Frenet-frame (moving frame).

Summary of the activity

This activity addresses vector functions. Many vector quantities in physics are the function of a parameter. For example, when an object is travelling through space, its vector position is a function of the time parameter $t$, and the vector $\vec{r}(t)$ that localizes the position of this travelling object is a vector function. Once you have completed this activity, you will be familiar with the notion of vector function as well as its derivation and integration.
Resources


Useful links


Description of the activity

This activity unfolds in a series of steps in order to help you master the objectives: vector function, vector function derivation and vector function integration. For greater efficiency, you are advised to tackle these sections one by one in order of presentation.
Formative assessment

Learners are required to work in collaboration to complete the seven exercises. Exercises 1 to 4 count for 10% each of the total mark, and exercises 5 to 7 count for 20% each.

Exercise 1

Given the vector function: \( \mathbf{r}(t) = (a + bt^2)\mathbf{i} + ct\mathbf{j} + (t^3 + 1)\mathbf{k} \), where \( a, b, c \) are positive constants, which of the following answers corresponds to the first and second derivatives of the vector \( \mathbf{r}(t) \) with respect to \( t \)?

A- \( 2b\mathbf{i} + ct \mathbf{j} + 3t^2 \mathbf{k} \)
B- \( 2b\mathbf{i} + 6t\mathbf{k} \)
C- \( 2b\mathbf{i} + c \mathbf{j} + 3t^2 \mathbf{k} \)
D- \( 2b\mathbf{i} + ct \mathbf{j} + 3t^2 \mathbf{k} \)
E- \( b\mathbf{i} + c \mathbf{j} + 6t\mathbf{k} \)
F- \( 2b\mathbf{i} + c \mathbf{j} \)

Exercise 2

In the Oxy plane, a unit vector: \( \mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \), where \( \theta \) is the angle of direction \((\mathbf{i}, \mathbf{u})\). The unit vector that is directly perpendicular to it is represented by the equation: \( \mathbf{v} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \). Assuming that the angle \( \theta \) is a function \( \theta(t) \) of the parameter \( t \), which of the following answers corresponds to \( \frac{d\mathbf{v}}{dt} \) and \( \frac{d^2\mathbf{v}}{dt^2} \)?

A - \( \mathbf{u} \)
B - \( -\mathbf{u} \)
C - \( -\frac{d\theta}{dt} \mathbf{u} \)
D - \( -\frac{d\theta}{dt} \mathbf{v} - \mathbf{u} \frac{d^2\theta}{dt^2} \)
Exercise 3

A material particle M travels in a circular motion in the $Oxy$ plane with an angular velocity $\omega = \omega_0 \vec{k}$, where $\omega_0$ is a constant. The velocity vector of the particle is given by the vector field: $\vec{v}(t) = \omega \wedge \vec{r}(t)$. Which of the following expressions corresponds to the acceleration vector of the particle?

A. $\frac{d\omega}{dt} \wedge \vec{r} + \omega \wedge \frac{d\vec{r}}{dt}$
B. $\omega \wedge \frac{d\vec{r}}{dt}$
C. $\omega \wedge \frac{d^2\vec{r}}{dt^2}$
D. None of the above

Exercise 4

An object M travels in the $Oxy$ plane with acceleration $\vec{a} = \alpha t^2 \vec{i} + (\beta - \gamma t) \vec{j}$, where $t$ is the time and $\alpha, \beta, \gamma$ are positive constants. At the initial time $t = 0$, the object is located at O with a null initial velocity. Which of the following answers corresponds to the velocity and position vectors of the travelling object?

A. $\frac{1}{3} \alpha t^3 \vec{i} + (\beta t - \frac{1}{2} \gamma t^2) \vec{j}$
B. $\frac{1}{3} \alpha t^3 \vec{i} + (\beta t - \frac{1}{2} \gamma t^2) \vec{j}$
C. $\frac{1}{12} \alpha t^4 \vec{i} + (\frac{1}{2} \beta t^2 - \frac{1}{6} \gamma t^3) \vec{j}$
D. $\alpha t^4 \vec{i} + (\frac{1}{2} \beta t^2 - \frac{1}{4} \gamma t^3) \vec{j}$
Exercise 5

In the orthonormal space, given the curve (C) of the following parametric equations:

\[ x = a \cos t \]
\[ y = a \sin t \]
\[ z = h t \]

where \( a \) and \( h \) are positive constants,

(a) Which of the following answers corresponds to the unit vector tangent to the curve, the radius of curvature and the principal normal vector?

A - \( \cos t \hat{i} + \sin t \hat{j} \)
B - \( -\cos t \hat{i} - \sin t \hat{j} \)
C - \( \frac{a \cos t}{\sqrt{a^2 + h^2}} \hat{i} - \frac{a \sin t}{\sqrt{a^2 + h^2}} \hat{j} \)
D - \( -a \sin t \hat{i} + a \cos t \hat{j} + \hat{k} \)
E - \( \frac{a \sin t}{\sqrt{a^2 + h^2}} \hat{i} + \frac{a \cos t}{\sqrt{a^2 + h^2}} \hat{j} + \frac{h}{\sqrt{a^2 + h^2}} \hat{k} \)
F - \( \frac{a}{a^2 + h^2} \)
G - \( a + \frac{h^2}{a} \)
H - \( \frac{a^2 + h^2}{h} \)

(b) The tangent to the curve forms a constant angle with the \( Oz \) axis. Is this statement correct?

A - TRUE
B - FALSE
Exercise 6

a) We have \( r, \theta, \text{ and } \varphi \) as the spherical coordinates of a point \( M \) in the space defined by \( r = \| \overrightarrow{OM} \| \), \( \theta = (\overrightarrow{Oz}, \overrightarrow{OM}) \) and \( \varphi = (\overrightarrow{Ox}, \overrightarrow{Om}) \), where \( m \) is the projection of \( M \) in the \( Oxy \) plane. The base unit vectors of the spherical coordinates are denoted by \( \mathbf{e}_r, \mathbf{e}_\theta, \text{ and } \mathbf{e}_\varphi \). Express the components of the base vectors as a function of the spherical coordinates, then the first partial derivatives of the base vectors. What is your conclusion?

b) In a ground frame of reference of a moving fluid mass whose velocity field is given by \( \mathbf{v} = x^2 \mathbf{i} - 2xy \mathbf{j} + 2t^2 \mathbf{k} \text{ m/s}^{-1} \), determine the acceleration vector of the fluid mass at the point with coordinates \((2,1,-4)\), in metres, at time \( t = 2 \text{ s} \).

Exercise 7

In the orthonormal space, the surface (\( S \)) of the Cartesian equation is \( z = x^2 + y^2 \)

a) Which of the following answers corresponds to the parametric representation of the surface (\( S \))?

A- \( \mathbf{r}(u, v) = (u + v) \mathbf{i} + uv \mathbf{j} + (u^2 + v^2) \mathbf{k} \)
B- \( \mathbf{r}(u, v) = ui + vj + (u^2 + v^2)k \)
C- \( \mathbf{r}(u, v) = (u^2 + v^2) \mathbf{i} + uv \mathbf{j} + (u^2 + v^2) \mathbf{k} \)
D- \( \mathbf{r}(u, v) = u^2 \mathbf{i} + v^2 \mathbf{j} + (uv) \mathbf{k} \)

b) Which of the following answers corresponds to the equation of the tangent plane to the surface (\( S \)) at a point \( M \) distinct from the origin?

A - \( 2(u + v)X - 2(u + v)Y + Z + (u + v)^2 = 0 \)
B - \( -2(x + y)X + 2Y + Z + (x + y)^2 = 0 \)
C - \( -2xX + x^2Y + Z = 0 \)
D - \( 2xX + 2yY - Z - z = 0 \)
Learning activities

- The learner must duly complete the following steps:

**Step 1**: Objectives to achieve (time: 10 minutes)
  - First, carefully read the objectives to achieve.
  - Write these objectives in your notebook and highlight or underline the key points that you must focus on.

**Step 2**: Obtain the required background knowledge by doing the required reading and visiting the Internet sites. This individual learning enables you to acquire the knowledge you need to achieve the objectives (time: 10 hours, using good time management).
  - Carefully read the chapter that covers these objectives (required reading).
  - In your notebook, write the key points of the course that you must focus on.
  - Consult the suggested references and visit the sites listed below in the Useful Links section.

**Step 3**: Understanding the course (group learning) (time: 5 hours)
  - Under your tutor’s guidance, share information using **chat** to make sure that you understand the material.
  - Work in **collaboration** with a group organized by your tutor.
  - Comply with the order and resolution time of the exercises, as indicated by your tutor.

**Step 4**: Assessment (time: 10 hours)
  - Each group will select a reporter, who will send the course instructor an email with an attached file containing the group’s report. This report will include the names of all the group members and the solutions to each exercise. The reporter may change from one exercise to another.
Answer key

Solution to exercise 1

The first derivative of the vector function is given by: \( \frac{d \vec{r}}{dt} = 2bi + cj + 3t^2 \vec{k} \), which corresponds to case C.

And the second derivative is given by: \( \frac{d^2 \vec{r}}{dt^2} = 2bi + 6t \vec{k} \), which corresponds to case B.

To learn more and reinforce what you have learned, review the definition of the derivative of a vector function and the properties of a derivative.

Solution to exercise 2

We must use the property of the derivative of the composed function \( \vec{v} \left[ \theta(t) \right] \), written as: \( \frac{d \vec{v}}{dt} = \frac{d \vec{v}}{d \theta} \frac{d \theta}{dt} \), hence we have \( \frac{d \vec{v}}{d \theta} = -\vec{u} \), which obtains:

\[ \frac{d \vec{v}}{dt} = -\vec{u} \frac{d \theta}{dt} \]

which corresponds to case C.

The second derivative is given by:

\[ \frac{d^2 \vec{v}}{dt^2} = \frac{d}{dt} \left( -\vec{u} \frac{d \theta}{dt} \right) = -\frac{d \vec{u}}{dt} \frac{d \theta}{dt} - \vec{u} \frac{d^2 \theta}{dt^2} \]

with

\[ \frac{d \vec{u}}{dt} = \frac{d \vec{u}}{d \theta} \frac{d \theta}{dt} = \vec{v} \frac{d \theta}{dt} \]

and finally \( \frac{d^2 \vec{v}}{dt^2} = -\vec{v} \left( \frac{d \theta}{dt} \right)^2 - \vec{u} \frac{d^2 \theta}{dt^2} \), which corresponds to case D.

To learn more and reinforce what you have learned, review the definition of the derivative of a turning unit vector.
Solution to exercise 3

By definition, the derivative of the velocity vector with respect to time $t$ gives the vector acceleration, as follows:

$$
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} (\omega \times \mathbf{r}) = \frac{d\omega}{dt} \times \mathbf{r} + \omega \times \frac{d\mathbf{r}}{dt} = \omega \times \frac{d\mathbf{r}}{dt}.
$$

Then, $\omega$ is a constant vector. Therefore, the answer is case B.

You can verify this result by calculating the second derivative of the position vector $\mathbf{r}(t)$, with the Cartesian components: $(R \cos \omega_0 t, R \sin \omega_0 t)$, from which we deduce that:

velocity vector is

$$
\frac{d\mathbf{r}}{dt} = -R \omega_0 \sin \omega_0 t \mathbf{i} + R \omega_0 \cos \omega_0 t \mathbf{j}.
$$

Finally, the acceleration vector is

$$
\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = R \omega_0^2 \cos \omega_0 t \mathbf{i} - R \omega_0^2 \sin \omega_0 t \mathbf{j}
$$

$$
\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = R \omega_0^2 \cos \omega_0 t \mathbf{i} - R \omega_0^2 \sin \omega_0 t \mathbf{j}
$$

Compare with

$$
\mathbf{a} = \mathbf{\omega} \times \mathbf{\omega} \times \mathbf{r} = \mathbf{\omega} \times \mathbf{v}
$$

$$
\mathbf{a} = \\
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 0 & \omega_0 \\
-R \omega_0 \sin \omega_0 t & R \omega_0 \cos \omega_0 t & 0
\end{vmatrix}
$$
Solution to exercise 4

The velocity vector is related to the acceleration vector by the equation:

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} \Rightarrow d\mathbf{v} = \mathbf{a} \, dt \]

And since \( \mathbf{v} = \int \mathbf{a} \, dt + \mathbf{R} \, \mathbf{v} = \int \mathbf{a} \, dt + \mathbf{R} \). In general this integration (taking into account the initial conditions, \( \mathbf{R} = 0, \mathbf{R} = 0 \)) gives:

\[ \mathbf{v} = \frac{1}{3} \alpha t^3 \mathbf{i} + (\beta t - \frac{1}{2} \gamma t^2) \mathbf{j} \]

A second integration leads to the position vector:

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} \Rightarrow d\mathbf{r} = \mathbf{v} \, dt \], and consequently \( \mathbf{r} = \int \mathbf{v} \, dt + \mathbf{C} \, \mathbf{r} = \int \mathbf{v} \, dt + \mathbf{C} \) and taking into account the initial position of the travelling object, i.e., \( \mathbf{C} = 0, \mathbf{C} = 0 \), we obtain:

\[ \mathbf{r} = \frac{1}{12} \alpha t^4 \mathbf{i} + (\frac{1}{2} \beta t^2 - \frac{1}{6} \gamma t^3) \mathbf{j} \]

Case A corresponds to the velocity and case C corresponds to the position vector.

Solution to exercise 5

(a)

The curve C has a radius vector \( \mathbf{r} \) of the form

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = a \cos t \mathbf{i} + a \sin t \mathbf{j} + h \mathbf{k} \]

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = a \cos t \mathbf{i} + a \sin t \mathbf{j} + h \mathbf{k} \]

At each point on the curve, the direction vector of the tangent is given by:

\[ \frac{d\mathbf{r}}{dt} = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + h \mathbf{k} \]

which is tangent to C.
The modulus of this vector is:
\[ \left\| \frac{d\boldsymbol{r}}{dt} \right\| = \sqrt{\alpha^2 \sin^2 t + \alpha^2 \cos t + h^2} \]

\[ \left\| \frac{d\boldsymbol{\alpha}}{dt} \right\| = \sqrt{a^2 + h^2} = \frac{ds}{dt} \], where \( s \) is the curvilinear abscissa measured from a fixed origin.

The unit vector of the tangent is given by:
\[ \hat{T} = \frac{d\boldsymbol{r}}{ds} = \frac{dr}{ds} \frac{dt}{ds} = -\frac{a \sin t}{\sqrt{a^2 + h^2}} \hat{i} + \frac{a \cos t}{\sqrt{a^2 + h^2}} \hat{j} + \frac{h}{\sqrt{a^2 + h^2}} \hat{k}, \quad \text{.........*} \]

which corresponds to case E.

The unit vector of the principal normal \( \hat{\mathbf{N}} \) and the radius of curvature \( \rho \rho \) are related by the Serret-Frenet equation:
\[ \frac{d\hat{T}}{ds} = \frac{1}{\rho} \hat{\mathbf{N}}. \]

Hence, we have:
\[ \left\| \frac{d\hat{T}}{ds} \right\| = \frac{1}{\rho} \text{ with } \frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt} \frac{dt}{ds} \]

The derivative of the unit tangent vector (Eqn. *) with respect to the parameter \( t \) is given by:
\[ \frac{d\hat{T}}{dt} = -\frac{a \cos t}{\sqrt{a^2 + h^2}} \hat{i} - \frac{a \sin t}{\sqrt{a^2 + h^2}} \hat{j} \]

whence:
\[ \frac{d\hat{T}}{ds} = -\frac{a \cos t}{a^2 + h^2} \hat{i} - \frac{a \sin t}{a^2 + h^2} \hat{j} \] and consequently \( \frac{1}{\rho} = \frac{a}{a^2 + h^2} \), thus
\[ \rho = a + \frac{h^2}{a}, \] which corresponds to case G.
Finally, we have: \( \overrightarrow{NN} \) as a unit vector in the direction \( \frac{d\overrightarrow{T}d\overrightarrow{T}}{ds ds} \). Hence
\[
\overrightarrow{N} = -\cos t \overrightarrow{i} - \sin t \overrightarrow{j},
\]
which corresponds to case B.

(b)

From the Eqn (*) in (a), we get:

\[
\vec{k} \cdot \vec{T} = \cos \theta = \frac{h}{\sqrt{a^2 + h^2}}
\]
where \( \theta \) is the angle between the tangent and the Oz axis. Because the cosine value is constant, this angle \( \theta \) is also constant. The statement is therefore true (case A).

**Solution to exercise 6**

a) The equation for converting Cartesian coordinates to spherical coordinates is given by: 
\[
x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.
\]

The components of vector \( \vec{e}_r \) are readily deduced from the components of the vector \( \vec{OM} = \vec{r} \) using a simple division by \( r \), as follows:
\[
\vec{e}_r = \sin \theta \cos \varphi \overrightarrow{i} + \sin \theta \sin \varphi \overrightarrow{j} + \cos \theta \overrightarrow{k}.
\]

The vector \( \vec{e}_r \) is perpendicular to the vector \( \vec{e}_r \); it is therefore contained in the plane \( (Oz, \vec{OM}) \) and forms an angle \( \theta + \frac{\pi}{2} \) with the Oz axis, whence:
\[
\vec{e}_\theta = \cos \theta \cos \varphi \overrightarrow{i} + \cos \theta \sin \varphi \overrightarrow{j} - \sin \theta \overrightarrow{k}.
\]

The vector \( \vec{e}_\varphi \) is in the Oxy plane, and forms an angle \( \varphi + \frac{\pi}{2} \) with the Ox axis. Consequently, we have:
\[
\vec{e}_\varphi = -\sin \varphi \overrightarrow{i} + \cos \varphi \overrightarrow{j}.
\]
Thus, only the vectors $\dot{e}_r$ and $\dot{e}_\theta$ are vector functions of two variables, the first partial derivatives of which are given by:

$$\frac{\partial \dot{e}_r}{\partial \theta} = \cos \theta \cos \varphi \hat{i} + \cos \theta \sin \varphi \hat{j} - \sin \theta \hat{k} = e_\theta$$

$$\frac{\partial \dot{e}_r}{\partial \varphi} = -\sin \theta \sin \varphi \hat{i} + \sin \theta \cos \varphi \hat{j} = \sin \theta e_\varphi$$

$$\frac{\partial \dot{e}_\theta}{\partial \theta} = -\sin \theta \cos \varphi \hat{i} - \sin \theta \sin \varphi \hat{j} - \cos \theta \hat{k} = -e_r$$

$$\frac{\partial \dot{e}_\theta}{\partial \varphi} = -\cos \theta \sin \varphi \hat{i} + \cos \theta \cos \varphi \hat{j} = \cos \theta e_\varphi$$

In conclusion we note that all the partial derivatives are orthogonal to the derived unit vector.

b) The acceleration vector is given by:

$$\mathbf{a} = \frac{d^2 \mathbf{v}}{dt^2} = \frac{\partial \dot{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \dot{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \dot{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \dot{v}}{\partial t} = \dot{v} \frac{\partial v}{\partial x} + \dot{v} \frac{\partial v}{\partial y} + \dot{v} \frac{\partial v}{\partial z} + \frac{\partial \dot{v}}{\partial t}$$

$$= x^2 (2 \dot{x} \hat{i} - 2y \dot{j}) + (-2xy)(-2x \dot{j}) + 2t^2 \hat{0} + 4tk = 2x^3 \hat{i} + 2x^2 y \hat{j} + 4tk$$

At point (2,1,-4) at time $t = 2s$, the acceleration is therefore:

$$\mathbf{a} = 16\hat{i} + 8\hat{j} + 8\hat{k} \text{ m/s}^2$$
Solution to exercise 7

(a)
The parametric representation of a surface (S) is given by the set of points M such that: \( \overrightarrow{OM} = \vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k} \). Specifically, if \( x = u \), \( y = v \), we have: \( \overrightarrow{OM} = x\vec{i} + y\vec{j} + z(x, y)\vec{k} \) and \( z = g(x, y) \) as the Cartesian equation for the surface (S), which can also be represented by the implicit function, with the three coordinates: \( f(x, y, z) = 0 \). Thus, the surface (S) given here may be represented parametrically by: \( \overrightarrow{OM} = \vec{r}(u, v) = u\vec{i} + v\vec{j} + (u^2 + v^2)\vec{k} \) which corresponds to case B.

(b)
The plane containing the points \( M(u, v) \), with direction vectors \( \frac{\partial\vec{r}}{\partial u} \) and \( \frac{\partial\vec{r}}{\partial v} \) is called the tangent plane at \( M \) to the surface (S). Given the point \( P(X, Y, Z) \) in the tangent plane, the equation of the tangent plane is then written as:

\[
\begin{bmatrix}
\frac{\partial\vec{r}}{\partial u} & \frac{\partial\vec{r}}{\partial v}
\end{bmatrix}
\begin{bmatrix}
X - x \\
y - y \\
z - z
\end{bmatrix}
= 0
\]

In other words:

\[
\begin{vmatrix}
X - x & Y - y & Z - z \\
x_u & y_u & z_u \\
x_v & y_v & z_v
\end{vmatrix}
= 0
\]

The equation of the tangent plane is therefore written as:

\[
\begin{vmatrix}
X - x & Y - y & Z - z \\
1 & 0 & 2x_0 \\
0 & 1 & 2y_0
\end{vmatrix}
= 0
\]

because we have:
\[ \frac{\partial \vec{r}}{\partial u} = \vec{i} + 2u \vec{k} \text{ and } \frac{\partial \vec{r}}{\partial v} = \vec{j} + 2v \vec{k} \text{ with } u = x \text{ and } v = y \]

After development, this gives:
\[ 2x \vec{X} + 2y \vec{Y} - Z - z = 0, \] which corresponds to case D.

Note that the equation of the tangent plane at M to the surface (S) can be written vectorially:
\[ \nabla f \cdot \overrightarrow{MP} = 0, \] because this plane is distorted locally at the surface when it is slightly rotated around point M and the surface considered corresponds to the level surface \( f = 0 \) of the function \( f \).

**Self-assessment**

Learners should make a note of their difficulties and errors when seeking the solutions to these exercises in order to avoid repeating them. They should review the areas of the course that they have not completely understood and prepare a summary assessment.

**Instructor’s guide**

The instructor corrects the group reports. He or she enters the corrections to the exercise answers in a work space that the learners can access. Corrections should be accompanied by adequate feedback on the errors made in the reports. The mark for the group is assigned to all group members, and counts for 20% of the final mark for the module.
Learning activity 3

Title of the activity: Physical fields and their derivatives

Learning time

30 hours

Guidelines: If you score at least 75% on this activity, you have done a good job, and you can continue.

If you score less than 50%, you should review the suggested readings and re-do the activity.

If you score between 50% and 75%, you have worked hard, but you must re-double your efforts in future.

Specific objectives

Upon completion of this activity, the learner should be able to:

- Define a scalar field and a vector field.
- Define a level curve, a level surface and field lines.
- Determine equations that define these notions and solve simple cases.
- Define the gradient of a scalar field.
- Calculate the gradient of a scalar function.
- Define the divergence and rotational of a vector field.
- Calculate the divergence and rotational of a vector field.
- Define the nabla (or del) operator.
- Determine the conditions for a vector field to be a gradient field.

Summary of the activity

This activity deals with physical fields. Many physical quantities are defined locally, meaning that they are defined at each point in a space-time domain. For example, an electric charge situated at a point in space creates a surrounding electrostatic field, which may be derived from a scalar function called the electrostatic potential. We know that the electrostatic field and the electrostatic potential are only functions of a given point in the space where they are determined. Once you have completed this activity, you will be familiar with the notions of scalar and vector fields.
Required reading


Useful resources:


Useful links

http://en.wikipedia.org/wiki/Vector
http://www.chez.com/ceh/formules.htm [in French]
http://en.wikipedia.org/wiki/Stokes%27_theorem
http://en.wikipedia.org/wiki/Gauss%27s_law

Description of the activity

This activity comprises several mandatory exercises that are designed to help the learner understand the following topics: scalar field – vector field – gradient of a vector field – divergence of a vector field – rotational of a vector field.
Formative assessment

Learners must work in collaboration to complete all five exercises. Each exercise counts for 20% of the mark.

Exercises

Exercise 1. Check the correct answer.

In a three-dimensional space, given that the point M in the Oxy plane has the coordinates x and y, we define the function of the point as \( f(M) = ||\overrightarrow{OM}|| \).

The representative surface of the function is:

A- a paraboloid
B- a hyperboloid
C- a cone

2- The level curves are:

A- planes parallel to the Oxy plane
B- spheres with centre O and radius r
C- circles with centre O and radius c

Exercise 2

Given the vector field \( \vec{V}(x, y) = y\vec{i} - x\vec{j} \):

a) The value of this field at the coordinate point (-1,1) is:

A- \( \vec{i} \)
B- \( \vec{j} \)
C- \( \vec{i} - \vec{j} \)
D- \( \vec{i} + \vec{j} \)

b) How are the field lines arranged?

A- straight lines parallel to \( \vec{i} \)
B- straight lines parallel to \( \vec{j} \)
C- circles with centre O
D- straight lines passing through the origin O
Exercise 3

For the scalar function of a point M in the coordinated space (x,y,z) given by
\[ f(M) = r^2, \]
where \( r \) is the modulus of the position vector \( \overrightarrow{OM} = \vec{r} \), \( \nabla f \) is
given by:

A- \( x^2 + y^2 + z^2 \)

B- \( xi + yj + zk \)

C- \( x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{j} \)

D- \( 2\vec{r} \)

Exercise 4

For the vector field \( \overrightarrow{V}(\vec{r}) = \frac{K\vec{r}}{||\vec{r}||^3} \), where \( K \) is a constant, the divergence of this field is given by:

A- 3

B- -1

C- 0

D- \( x^2 + y^2 + z^2 \)

Exercise 5

Consider a material particle M that travels in a circular motion in the Oxy plane
with an angular velocity \( \vec{\omega} = \vec{\omega}_0 \vec{k} \), where \( \vec{\omega}_0 \) is a constant. The velocity vector
of the particle is given by the vector field \( \overrightarrow{V}(\vec{r}) = \vec{\omega} \times \vec{r} \).

a- The rotational of the vector field is given by:

A- 0

B- 0

C- \(-2\omega\)

D- \(2\omega\)

b- This vector field is a gradient field. Is this statement true or false?

A- TRUE

B- FALSE
Learning activities

- The learner must duly complete the following steps:

**Step 1:** Objectives to achieve (time: 10 minutes)
- First, carefully read the objectives to achieve.
- Write these objectives in your notebook and highlight or underline the key points that you must focus on.

**Step 2:** Obtain the required background knowledge by doing the required reading and visiting the Internet sites. This individual learning enables you to acquire the knowledge you need to achieve the objectives (time: 14 hours and 50 min, using good time management).
- Carefully read the chapter that covers these objectives (required reading).
- In your notebook, write the key points of the course that you must focus on.
- Consult the suggested references and visit the sites listed below in the Useful Links section.

**Step 3:** Understanding the course (group learning) (time: 5 hours)
- Under your tutor’s guidance, share information using chat to make sure that you understand the material.
- Work in collaboration with a group organized by your tutor.
- Comply with the order and resolution time of the exercises, as indicated by your tutor.

**Step 4:** Assessment (time: 10 hours)
- Each group will select a reporter, who will send the course instructor an email with an attached file containing the group’s report. This report will include the names of all the group members and the solutions to each exercise. The reporter may change from one exercise to another.
Answer key

Solution to exercise 1

1- To graphically represent a function of a single variable \( y = f(x) \), we situate it in the Oxy plane, and to each x variable we associate the point whose ordinate is equal to \( y = f(x) \). The geometric location of all these points in the definition domain of the function constitutes the representative curve (or graph) of the function \( f(x) \). This curve is therefore planar. Now consider a function of 2 variables. To represent it, we must situate in a three-dimensional space, such as \( z = f(x,y) \), with respect to the Oxyz axes. At each pair of points \((x,y)\) belonging to the Oxy plane, we associate a datum point \( z = f(x,y) \). The geometric location of all these points constitutes a graphic representation of the function \( f(x,y) \): generally, a surface in a three-dimensional space.

To represent the function \( f(x, y) = \sqrt{x^2 + y^2} \), we will first set up a double-entry table:

Table 1: Values of \( f(x, y) = \sqrt{x^2 + y^2} \)

<table>
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<tr>
<th>( y )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>4.12</td>
<td>4.47</td>
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</tbody>
</table>

Looking at the column corresponding to \( x = 0 \), with corresponds to the Oyz plane, if we map \( f(0,y) \), we see that there are two straight lines branching symmetrically from the Oz axis at a 45° angle. The same holds true for the map of \( f(x,0) \). If we fill in the space between these straight lines, we obtain a cone having a semi-angle of 45° which issues at the apex. The answer is therefore case C.
2- The level curves are defined by:

\[ z = f(x,y) = c, \text{ where } c = \text{constant} \]

Thus, \( x^2 + y^2 = c^2 \),

so we have circles with centre O and radius c, a radius that increases by one unit with each unit increase in z data.

Note that these level curves are projections of different circles constructed on the surface of the cone when sectioned by the plane \( z = \text{constant} \).

To learn more and to reinforce what you have learned, review the definition of a level curve of a scalar field.

**Solution to exercise 2**

a) Here, the vector field is two-dimensional, so we can represent it in the Oxy plane. We are situated at point \( M(-1,1) \), which is considered the origin of the vector field, written here as: \( \vec{V}(-1,1) = \vec{i} + \vec{j} \). This vector therefore has the components \((1,1)\) and modulus equal to \( \sqrt{2} \), so the answer is case D.

b) To represent a vector field in space, we consider various points \( M(x,y) \), then calculate the components of the vector field at these points. However, a simple method is to trace the field lines: a group of tangent lines at each point in the vector field. They are given by the equation:

\[ \vec{V} (x, y) \land d\vec{r} = 0 \text{ avec } d\vec{r} = d\vec{x}i + d\vec{y}j, \text{ soit :} \]

\[ (y\vec{i} - x\vec{j}) \land (d\vec{x}i + d\vec{y}j) = (xdx + ydy)k = 0 \]

Knowing that \( d(x^2 + y^2) = 0 \), and by integration, we obtain:

\[ x^2 + y^2 = \text{constant} \]

This is a group of circles with centre O, so the answer is therefore case C.
Solution to exercise 3

The gradient of a scalar function $f(x,y,z)$ is given by:

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

Thus:

$$\nabla f = 2xi + 2yj + 2zk = 2\vec{r}$$

The correct answer is therefore case D.

Solution to exercise 4

The divergence of a vector field is given by:

$$\text{div}\,\vec{V}(\vec{r}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Thus, by writing the vector field in the following form:

$$\vec{V}(\vec{r}) = \frac{Kx}{(x^2 + y^2 + z^2)^{3/2}}i + \frac{Ky}{(x^2 + y^2 + z^2)^{3/2}}j + \frac{Kz}{(x^2 + y^2 + z^2)^{3/2}}k$$

we obtain:

$$\frac{\partial V_x}{\partial x} = \frac{K}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3Kx^2}{(x^2 + y^2 + z^2)^{5/2}}$$

and

$$\frac{\partial V_y}{\partial y} = \frac{K}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3Ky^2}{(x^2 + y^2 + z^2)^{5/2}}$$

and

$$\frac{\partial V_z}{\partial z} = \frac{K}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3Kz^2}{(x^2 + y^2 + z^2)^{5/2}}$$
Whence, by summing these expressions, we obtain:

$$\text{div}\mathbf{V}(r) = \frac{3K}{(r^2 + y^2 + z^2)^{3/2}} - \frac{3K(r^2 + y^2 + z^2)}{(r^2 + y^2 + z^2)^{5/2}} = 0$$

The answer is therefore case C.

**Solution to exercise 5**

a) The rotational (or curl) of a vector field is given by:

$$\text{rot}\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k}$$

Here, the components of the vector field are given by:

$$V_x = -y\omega_0, \quad V_y = x\omega_0, \quad \text{et} \quad V_z = 0,$$

thus:

$$\text{rot}\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = 2\omega_0\mathbf{k}$$

The answer is case D.

b) A vector field is a gradient field if, and only if, \( \nabla \mathbf{\psi} \) is a scalar function \( \mathbf{f} \) such that \( \text{grad} \mathbf{f} = \mathbf{V} \) or if \( \text{rot}\mathbf{V} = \mathbf{0} \), which is not the case here. The answer is therefore case B.
Self-assessment

Learners should make a note of their difficulties and errors when seeking the solutions to these exercises in order to avoid repeating them. They should review the areas of the course that they have not completely understood and prepare a summary assessment.

Instructor’s guide

The instructor corrects the group reports. He or she enters the corrections to the exercise answers in a work space that the learners can access. Corrections should be accompanied by adequate feedback on the errors made in the reports. The mark for the group is assigned to all group members, and counts for 20% of the final mark for the module.
Learning activity 4

Title of the activity: Spatial integration – Applications in physics

Learning time

40 hours

Guidelines: If you score at least 75% on this activity, you have done a good job, and you should continue.

If you score less than 50%, you should review the suggested readings and re-do the activity.

If you score between 50% and 75%, you have worked hard, but you must re-double your efforts in future.

Specific objectives

At the end of this activity, you should be able to:

• Calculate a curvilinear integral by curve parameterization.
• Calculate the circulation of a vector field.
• Define the integral condition for a vector field to be a gradient field and its operation.
• Determine the derivation function for a gradient field using the circulation method.
• Calculate a double integral for any parameterizes surface
• Calculate a vector field flux.
• Calculate volume integrals.
• Geometrically interpret the nabla operator.
• Explain the notion of an electromagnetic field.
• Apply mathematical theories of integrals.
• Apply the local laws of electromagnetism.
• Describe wave propagation in media.
• Describe the phenomena of reflection, refraction, dispersion and absorption.
Summary of the activity

This activity covers line, surface and volume integrals. The physical fields that we studied previously can usually be associated with integrals along the lines, surfaces and volumes that play an important role in the field of physics. For example, determining the work of a force field as an object travels through it requires calculating the force field along the trajectory of the travelling object. By completing this activity, you will become familiar with spatial integrals.

Useful resources


Useful links

http://assocampus.ifrance.com/pages/raps.htm

[In French]


[In French]

Description of the activity

This activity unfolds in a series of steps in order to help the learner grasp the different themes: circulation of a vector field, vector field flux, volume integrals, Gauss’ theorem and Stokes’ theorem, and the application of these integral theorems in physics. For greater efficiency, you are advised to tackle these sections one by one in order of presentation:
Formative assessment

Learners must work in collaboration to complete all seven exercises.

Exercises 1 and 5 count for 15% each of the total mark, exercises 2, 3 and 4 count for 10% each, and exercises 6 and 7 count for 20% each.

Exercise 1

We want to calculate the circulation of a vector field \( \mathbf{E}(r) = \frac{Kr}{||r||^2} \), where \( K \) is a positive constant, along the paths connecting the origin at point B(1,2) of the Oxy plane, indicated as follows:

a) the straight line segment OB;

b) the path OAB, where A is a point with coordinates (1,0);

c) the parabola joining O and B.

Do we obtain the same result for any path OB? If yes, how do you explain this result?

Exercise 2

Calculate the flux of the vector field \( \mathbf{E}(r) = \frac{Kr}{||r||^2} \), where \( K \) is a positive constant, across a sphere with centre O and radius R. Which of the following answers is correct?

A- \( 4\pi k \)

B- 0

C- \( 4\pi R^3 \)

D- None of these

Exercise 3

Calculate the volume of a cone with axis Oz, with apex in O, height h and base radius R.

Which of the following answers is correct?

A - \( \pi R^2 h \)

B - \( \frac{\pi R^2 h}{3} \)

C - \( \frac{4}{3} \pi R^2 h \)

D - Autre réponse
Exercise 4

a) If $V$ is the enclosed volume of an enclosed surface $(S)$, show that

$$\int_S \nabla \cdot d\vec{S} = 3V$$

b) If $\vec{B} = \text{rot} \ \vec{A}$, show that $\int_S \vec{B} \cdot d\vec{S} = 0$ for the entire closed surface $(S)$.

Exercise 5

Verify Stokes’ theorem for $\vec{V}(x, y, z) = yi - xj + zk$, where the boundary $(C)$ of the surface $(S)$ is a circle in the $xy$ plane of the unit radius with centre at the origin.

Exercise 6

A fluid with volumic mass $\rho(x, y, z, t)$ travels at a velocity $\vec{v}(x, y, z, t)$ through a three-dimensional space. Assume that there are no mass sources or sinks. Show that the conservation of mass for this fluid is written as the differential equation:

$$\frac{\partial \rho}{\partial t} + \text{div} \ \vec{J} = 0$$

where $\vec{J} = \rho \vec{V}$.

Exercise 7

We have a separating surface of two homogeneous linear isotropic media. Assume a surface density charge $\sigma$ and a superficial density current $\vec{K}$. Consider the case of a stationary regime, where $\rho$ and $\vec{J}$ denote the volumetric densities of the charge and the current, respectively, as the electromagnetic field sources. Show that the conditions of passage imposed on the electrical induction field $\vec{D}$ and magnetic excitation $\vec{H}$ with corresponding subscript (1 and 2) are written as:

$$(\vec{D}_2 - \vec{D}_1) \cdot \vec{n} = \sigma$$

$$\vec{n} \wedge (\vec{H}_2 - \vec{H}_1) = \vec{K}$$

where $\vec{n}$ is the unit normal vector to the separating surface in the direction from medium 1 to medium 2.
**Learning activities**

- The learner must duly complete the following steps:

**Step 1:** Objectives to achieve (time: 10 minutes)
  - First, carefully read the objectives to achieve.
  - Write these objectives in your notebook and highlight or underline the key points that you must focus on.

**Step 2:** Obtain the required background knowledge by doing the required reading and visiting the Internet sites. This individual learning enables you to acquire the knowledge you need to achieve the objectives (time: 20 hours, using good time management).
  - Carefully read the chapter that covers these objectives (required reading).
  - In your notebook, write the key points of the course that you must focus on.
  - Consult the suggested references and visit the sites listed below in the Useful Links section.

**Step 3:** Understanding the course (group learning) (time: 5 hours)
  - Under your tutor’s guidance, share information using chat to make sure that you understand the material.
  - Work in collaboration with a group organized by your tutor.
  - Comply with the order and resolution time of the exercises, as indicated by your tutor.

**Step 4:** Assessment (time: 15 hours)
  - Each group will select a reporter, who will send the course instructor an email with an attached file containing the group’s report. This report will include the names of all the group members and the solutions to each exercise. The reporter may change from one exercise to another.
Answer key

Solution to exercise 1

The elementary vector displacement is written as: $d\vec{r} = dx\vec{i} + dy\vec{j}$ and the integral along a curve (i.e., path integral or circulation) is given by: $\Gamma = \int_C E \cdot d\vec{r}$.

We will therefore determine this integral along the indicated curves.

a) Along the right-hand segment OB:

Since the equation for the right-hand OB is given by: $y = 2x \Rightarrow dy = 2dx$

we then have:

$$C = \int_{\Gamma_1} -k(xdx + ydy) = -5k \int_{\Gamma_1} xdx = -5k \int_0^1 xdx$$

and finally:

$$C = -\frac{5k}{2}$$

b) This path is decomposed into two:

- the path OA, where the displacement vector is reduced to: $d\vec{r} = dx\vec{i}$, thus:

$$C_1 = -k \int_{\Gamma_2} xdx = -k \int_0^1 xdx = -\frac{k}{2}$$

- the path AB, where the displacement vector is reduced to: $d\vec{r} = dy\vec{j}$, thus:

$$C_2 = -k \int_{\Gamma_2} ydy = -k \int_0^2 ydy = -2k$$

whence $C = C_1 + C_2 = -\frac{5k}{2}$

c) The equation of the parabola is given by: $y = 2x^2 \Rightarrow dy = 4xdx$
Consequently, \[ C = \int_{C} -k(\text{d}x + \text{d}y) = -k \int_{C} (x + 8x^3) \text{d}x = -k \left[ \frac{x^2}{2} + 2x^4 \right]_0. \]

Whence: \[ C = -\frac{5k}{2}. \]

In conclusion, the circulation of this field vector is independent of the path followed; it depends on the initial and final point only. This vector field is therefore a gradient field (or central field).

To learn more and to reinforce what you have learned, review the definition of the circulation of a field vector. Verify that the field is a gradient field by calculating its rotational.

**Solution to exercise 2**

The surface element of the sphere is given by \[ d\mathbf{S} = dS \mathbf{e}_r, \] where \( \mathbf{e}_r \) is the unit vector of the base of the sphere (the surface element is perpendicular to the radius of the sphere). It becomes:

\[ \Phi = \int \frac{k\mathbf{r}}{||r||^3} \cdot dS \mathbf{e}_r. \]

And because on the sphere we have a constant radius R, we therefore obtain:

\[ \mathbf{r} = R\hat{r} \Rightarrow R^2 = \mathbf{r}^2 \Rightarrow |\mathbf{r}|^3 = R^3 \Rightarrow |\mathbf{r}|^3 = R^3 \]

\[ \Phi = \int \frac{k\mathbf{r}}{||r||^3} \cdot dS \mathbf{e}_r = \frac{k}{R^2} \int_{\text{sphere}} dS = 4\pi k \] which is case A.

To learn more and to reinforce what you have learned, review the definition of a vector field flux.
Solution to exercise 3

We would like to use cylindrical coordinates. The volume element $dV$ is given by:

$$dV = \rho \, d\rho \, d\theta \, dz,$$

with the integral domain:

$$0 \leq z \leq h; \quad 0 \leq \theta \leq 2\pi; \quad 0 \leq \rho \leq r(z).$$

For a given value of $z$, the radius is given by $r(z) = \frac{R_z}{h}$: the integrals at $r$ and $z$ therefore cannot be separated, although we must begin with the integral at $r$:

$$V = \int_0^{R_z/h} \int_0^{2\pi} \int_0^h \rho \, d\rho \, d\theta \, dz = \int_0^{2\pi} d\theta \int_0^h \left( \int_0^{R_z/h} \rho \, d\rho \right) dz$$

thus:

$$V = 2\pi \int_0^h \frac{R_z^2 z^2}{2h^2} dz = 2\pi \left[ \frac{R_z^2 z^3}{6h^2} \right]_0^h = \frac{\pi R_z^2 h}{3}.$$

Answer is case B.

Solution to exercise 4

Remember that Gauss’ theorem (or the divergence theorem) states that the surface integral of the normal component of a vector $\vec{V}$, extended over the whole closed surface (S) (i.e., the flux described by a vector field $\vec{V}$ across an closed surface (S)) is equal to the triple integral of the divergence of $\vec{V}$ for the whole closed surface of the volume.

a) From the divergence theorem, we have:

$$\iiint_S \vec{r} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{r} \, dV$$

$$= \iiint_V \left( \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) \, dV = \iiint_V 3 \, dV = 3V.$$
b) Again from the divergence theorem, we have:

$$\iiint_{V} \mathbf{\nabla} \cdot \mathbf{B} \, dV = \iiint_{S} \mathbf{\nabla} \cdot \mathbf{B} \, dS$$

Thus:

$$\mathbf{\nabla} \cdot \mathbf{B} = \mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{A}) = (\mathbf{\nabla} \times \mathbf{\nabla}) \cdot \mathbf{A} = 0$$

whence,

$$\iiint_{S} \mathbf{\nabla} \cdot \mathbf{B} \, dS = \iiint_{V} \mathbf{\nabla} \cdot \mathbf{B} \, dV = 0$$.

Recall that in electromagnetism, the magnetic field $\mathbf{B}$ derives from a vector potential $\mathbf{A}$, therefore:

$$\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}.$$ If so, then $\mathbf{\nabla} \cdot \mathbf{B} = 0$, and $\mathbf{B}$ is said to be a solenoidal field. In general, if $\mathbf{B}$ is a solenoidal field, there is a vector potential $\mathbf{A}$.

**Solution to exercise 5**

Recall that Stokes’ theorem states that the curvilinear integral

$$\oint_{C} \mathbf{V} \cdot d\mathbf{r} = \iiint_{S} (\mathbf{\nabla} \times \mathbf{V}) \cdot \mathbf{n} \, dS$$

of the tangential component of a vector $\mathbf{V}$ along a closed curve (C) is equal to the surface integral of the normal rotational component of $\mathbf{V}$ over the whole surface (S) with boundary (C).

Furthermore:

the circulation of a vector field $\mathbf{V}$ along a closed curve (C) is equal to the rotational flux of $\mathbf{V}$ over the whole surface (S) enclosed by (C).

In the xy plane, $z = 0$, where $\mathbf{V} = y\mathbf{i} - x\mathbf{j}$ and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$, such that the circulation of the vector field is:

$$\Gamma = \oint_{C} \mathbf{V} \cdot d\mathbf{r} = \oint_{C} y\, dx - x\, dy$$

Given $x = \cos \theta$, $y = \sin \theta$ with $0 \leq \theta < 2\pi$ for the parametric equations of (C), we then have:
\[ \Gamma = \oint_c \mathbf{V} \cdot d\mathbf{r} = \oint_c y \, dx - x \, dy = \int_0^{2\pi} - (\sin^2 \theta + \cos^2 \theta) \, d\theta = -2\pi \]

Furthermore:

\[
\text{rot}\mathbf{V} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & z
\end{vmatrix} = -2\mathbf{k}
\]

Thus: \[\int_S \text{rot}\mathbf{V} \cdot d\mathbf{S} = \int_S -2\mathbf{k} \cdot d\mathbf{S} = -2\pi.\]

Therefore, Stokes’ theorem is verified.

**Solution to exercise 6**

\( V \) is a volume of fluid enclosed by a surface \( S \). The mass of fluid contained in this volume at any time is given by:

\[ m = \iiint_V \rho \, dV. \]

The fluid volume travelling across a surface element \( dS \) per unit of time is given by: \( \mathbf{v} \cdot \mathbf{n} \, dS \) (volume contained in a cylinder with base surface \( dS \) and height \( \mathbf{v} \) ). Thus, the fluid mass escape from volume \( V \) per unit of time is: \( \iiint_S \rho (\mathbf{v} \cdot \mathbf{n}) dS \).

According to the divergence theorem, we have:

\[ \iiint_S \rho \mathbf{v} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \rho \mathbf{v} \, dV. \]

The equation for the conservation of mass is therefore written as:

\[ \frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \iiint_V \rho \, dV = -\iiint_S \rho \mathbf{v} \cdot \mathbf{n} \, dS = -\iiint_V \nabla \cdot \rho \mathbf{v} \, dV. \]

Because this equation is valid for any given volume, we have:

\[ \frac{\partial \rho}{\partial t} + \text{div} \mathbf{J} = 0 \quad \text{where} \quad \mathbf{J} = \rho \mathbf{v}. \]
In physics, this equation is called an equation of continuity. It is also used in electromagnetics, where it is used to convert the equation of conservation of the electric charge when \( \rho \) is the volumetric density of the charge (charge density) and \( J = \rho v \) is the volumetric density of the current (current density).

For an incompressible fluid, \( \rho \) is constant and the velocity field is solenoidal, because we have \( \text{div} \, v = 0 \).

**Solution to exercise 7**

Recall that for a stationary regime (independent of time), the Maxwell–Gauss and Ampère–Maxwell equations are written as:

\[
\begin{align*}
\text{div} \, D &= \rho \\
\text{rot} \, H &= J
\end{align*}
\]

Below: Medium 2; Medium 1

Imagine an elemental cylindrical form with base surface \( dS \) (Fig. 1) and height \( dh \) (an infinitely small quantity of the second order compared to the \( dS \) dimension). Consider the flux of \( \vec{D} \) over the cylinder surface.

- The flux over the lateral surface is negligible, because \( dh \) is assumed at the second order.

**Fig 1: Elemental cylindrical form**

- The flux over the base surfaces, which is equal to the total charge present in the cylinder, is:
\[ D_2 \cdot \mathbf{n} \, dS - D_1 \cdot \mathbf{n} \, dS = \sigma \, dS \]

and after reduction or simplification,

\[ (D_2 - D_1) \cdot \mathbf{n} = \sigma \]

A discontinuity in the normal component of the electrical induction vector appears, equal to the superficial charge density.

Now imagine a simple rectangular circuit located on a plane normal to the surface of separation, with length \( L \) parallel to the tangent to the separation surface, of magnitude \( h \) (an infinitely small quantity of the second order compared to \( L \)) parallel to the normal at the surface of separation.

[Figure 2: Elemental rectangular circuit]

The circuit rotates in a positive direction around \( \mathbf{n}_b \) (Fig. 2).

Let us calculate the circulation of \( \mathbf{H} \) for this circuit. It includes:

- the circulation over the negligible \( h \) heights(or)amplitudes,
- the circulation over the lengths, which, according to Stokes’ and Ampère’s theorems, is equal to the intensity of the current that crosses the rectangle, which is:

\[
\oint \mathbf{H} \cdot d\mathbf{l} = \iint_S \text{rot}\, \mathbf{H} \cdot \mathbf{n}_b \, dS = \iint_S \mathbf{K} \cdot \mathbf{n}_b \, dS
\]

\[
= (\mathbf{H}_1 - \mathbf{H}_2) \cdot \mathbf{t} \, L = K \cdot \mathbf{n}_b \, L \quad \Rightarrow \quad (\mathbf{H}_1 - \mathbf{H}_2) \cdot \mathbf{t} \, L = K \cdot (\mathbf{t} \wedge \mathbf{n}) = \mathbf{t} \cdot (\mathbf{n} \wedge K)
\]

whence

\[
\mathbf{H}_1 - \mathbf{H}_2 = \mathbf{n} \wedge K
\]

and hence:

\[
\mathbf{n} \wedge (\mathbf{H}_2 - \mathbf{H}_1) = K
\]
A discontinuity in the component tangent to the magnetic excitation vector appears, equal to the superficial current density.

**Self-assessment**

Learners should make a note of their difficulties and errors when seeking the solutions to these exercises in order to avoid repeating them. They should review the areas of the course that they have not completely understood and prepare a summary assessment.

**Instructor’s guide**

The instructor corrects the group reports. He or she enters the corrections to the exercise answers in a **workspace that the learners can access**. Corrections should be accompanied by adequate feedback on the errors made in the reports. The mark for the group is assigned to all group members, and counts for 20% of the final mark for the module.
XI. List of key concepts (Glossary)

1. **Vector**: a directional quantity that is characterized by two entities, its direction (or sense) and intensity (or modulus).

2. **Function**: a mathematical function is a relationship whereby a quantity \( x \) belonging to a set \( A \) (departure) is associated with another quantity belonging to a set \( B \) (arrival).

3. **Field**: a (scalar) vector field is defined as a construction in vector calculus that associates a vector or scalar to every point in a space. This quantity is called a local quantity (or position quantity).

4. **Field line**: a curve such that each of its points is tangent to the field vector defined at that point. In the particular case where the vector field is a force field, the term **force line** is used.

5. **Level surface (or level curve)**: a set of points for which the scalar field has a fixed value.

6. **Circulation**: A circulation (path integral) \( C \) of vector field \( \vec{A} \) along a curve \( \Gamma \) is defined by the curvilinear integral: 

   \[
   C = \oint_{\Gamma} \vec{A} \cdot d\vec{l},
   \]

   where \( d\vec{l} \) is an infinitely small directed element of the curve.

7. **Flow (or) Flux**: The flux \( \Phi \) of a vector field \( \vec{A} \) over a surface \( S \) is given by the surface integral:

   \[
   \Phi = \iint_{S} \vec{A} \cdot \hat{n} \, dS,
   \]

   where \( \hat{n} \) is the positive unit vector normal to the infinitely small surface element \( dS \).

8. **Operator**: A mathematical symbol that denotes an operation to perform. The **nabla** operator (also called del) is a vector derivative operator denoted by \( \nabla \), and has no meaning in itself. It must be applied to a scalar of vector point function, and it has both vector and derivative properties.

9. **Gradient**: When the vector derivative operator \( \nabla \) acts on the scalar function of point \( f(M) \), it defines the gradient of this function and represents the rate of variation of this function along a given direction (directional derivative).

10. **Potential**: The name given to a scalar function of point \( f(M) \) in the particular case of a vector field \( \vec{V}(M) \), in which \( \vec{V}(M) = \nabla f(M) \).
XI. List of required reading

Reading No. 1


Summary:
Vector algebra in R$^3$

This chapter discusses vectors. It begins with the basic notions and proceeds to develop common algebraic operations with vectors: addition, of vector multiplication, of a vector by a scalar, scalar (dot) product, vector (cross) product and mixed product.

Purpose:
This reading provides the learner with the information required to solve the exercises presented in Learning activity 1 and some of the exercises in the Summary assessment. In addition, various operations with vectors are used throughout the module. Therefore, the learner is strongly advised to read this chapter carefully.

Reading No. 2


Summary:
This chapter introduces an essential notion in mechanics and electromagnetism: the vector function. It begins with the notions, of vector components, then proceeds to develop the formulations of vector derivatives and integrals. Finally, it demonstrates how these functions can be used to define the parameters of curves and surfaces in space.

Purpose:
This reading provides the learner with the information required to solve the exercises presented in Learning activity 3 and some of the exercises in the Summary assessment. In addition, the various notions concerning curves and surfaces are used later on in the module. We therefore advise the learner to study them carefully.
Reading No. 3

**Title:** YOUM, I. (2006). Les champs physiques. Université Cheikh Anta DIOP de Dakar. Cours inédit

**Summary:**
This chapter introduces the general notion of a scalar or vector function of a point in space. The notion of field plays a very important role, even in elementary physics. Using derivative operations in a field, the notion of the nabla operator, a derivative vector operator, is introduced.

**Purpose:**
This reading provides the learner with the information required to solve the exercises presented in Learning activity 3 and some of the exercises in the Summary assessment. We strongly advise the learner to study these carefully, given the importance of this subject in physics.

Reading No. 4

**Title:** YOUM, I. (2006). Intégrales spatiales. Université Cheikh Anta DIOP de Dakar. Cours inédit

**Summary:**
This chapter introduces the notion of the integral operations most commonly applied to local scalar and vector quantities. The outcome of these operations is that local quantities disappear to the benefit of the overall properties of a physical system. This enables the establishment of the physical significance of the derivative vector operator (nabla).

**Purpose:**
This reading provides the learner with the information required to solve the exercises presented in Learning activity 4 and some of the exercises in the Summary assessment.
Reading No. 5


Summary:
This chapter examines electromagnetic wave propagation in linear and isotropic media (beginning with a review of wave propagation in a vacuum). These media can be either dielectric or conducting. The Snell–Descartes laws of electromagnetism are then established by electromagnetic theory. The Fresnel coefficients are also established.

Purpose:
This reading provides the learner with a number of opportunities to apply vector analysis, as learned in the previous chapters. It also provides the information required to solve the exercises presented in Learning activity 4 and some of the exercises in the Summary assessment.
XII. List of useful links

Useful link No. 1

URL title: http://tanopah.jo.free.fr/seconde/vecteursalpha.php [in French]
http://en.wikipedia.org/wiki/Vector ]

Screenshot [in French]

Description
This link describes vectors, including definitions, characteristics and some operations.

Purpose
This link provides information on vectors, their representation and some operations.
Useful link No. 2

URL title: [http://fr.wikipedia.org/wiki/Vecteur#Op%C3%A9rations_sur_les_vecteurs][in French]
http://en.wikipedia.org/wiki/Vector_space

Screenshot

Description

This article defines the vector as a mathematical object and its usefulness in representing physical quantities.

Purpose

This article provides the learner with information on the uses of vectors in mathematics, physics and computing, as well as various basic vector operations.
Useful link No. 3

**URL title:** [http://fr.wikipedia.org/wiki/Produit_scalaire](http://fr.wikipedia.org/wiki/Produit_scalaire) [in French]


**and Scalar Product of Vectors:** [http://hyperphysics.phy-astr.gsu.edu/Hbase/vscal.htm](http://hyperphysics.phy-astr.gsu.edu/Hbase/vscal.htm)

**Screenshot**

These articles [i.e., the two English articles] describe the scalar product of two vectors, including definitions, notation, scalar product in real space and scalar product in a complex space.

**Purpose**

These sites enable the learner to find out more about scalar products and grasp the underlying theoretical aspects.
Useful link No. 4

URL title: http://jpm-chabert.club.fr/maths/Lexique/vecteur.htm [Not found]


Screenshot

Description
This link provides a definition of vector and several operations.

Purpose
This link provides the learner with the definition and characteristics of vectors.
Useful link No. 5

URL title:

Screenshot

Description
This link provides definitions of the mixed product and some of its properties.

Purpose
This link provides the learner with information on mixed products and their various properties.
Useful link No. 6

URL title:
Product of Vectors, at http://www-math.mit.edu/~djk/18_022/chapter02/section02.html

Screenshot

Description
This link describes the double product of vectors and how it can be represented vectorially.

Purpose
This link provides the learner with additional information on the double vector product, which is a fundamental concept in this course.
Useful link No. 7

URL title: Vector Multiplication: http://physics.info/vector-multiplication/

Screenshot

Description
This link describes how the vector product is applied in physics, specifically for the moment of a force about a point.

Purpose
It provides the learner with information on certain applications of mathematics in physics, particularly vector products.
Useful link No. 8

URL title:  
http://www.chez.com/ceh/formules.htm [in French]  
http://en.wikipedia.org/wiki/Gradient  
and  
http://www.archive.org/stream/vectorcalculuswi00shawrich/vectorcalculuswi00shawrich_djvu.txt

Screenshot

Description

These links describe the Cartesian, cylindrical and spherical coordinates of a scalar gradient and a vector gradient. They also explain vector divergence, vector rotation, and the Laplacian operator of a scalar and a vector in terms of Cartesian, cylindrical and spherical coordinates.

Purpose

These links provide the learner with more information on the difference between scalar and gradient vectors in the three coordinate systems: Cartesian, cylindrical and spherical.
Link No. 9

URL title: http://fr.wikipedia.org/wiki/Th%C3%A9or%C3%A8me_d%27unicit%C3%A9_de_Stokes [in French]

and Dirichlet’s principle: http://en.wikipedia.org/wiki/Dirichlet%27s_principle
or http://www.owlinet.rice.edu/~fjones/chap13.pdf

Screenshot

Description
This link explains Stokes’ theorem, demonstrating its uniqueness, usefulness and limitations.

Purpose
This link provides the learner with more information about Stokes’ theorem and its uniqueness in order to solve some of the exercises.
Link No. 10

Divergence Theorem (or Flux-divergence theorem)

URL title: [http://fr.wikipedia.org/wiki/Th%C3%A9or%C3%A8me_de_flux-divergence](http://fr.wikipedia.org/wiki/Th%C3%A9or%C3%A8me_de_flux-divergence) [in French]


And [http://en.wikipedia.org/wiki/Maxwell%27s_equations](http://en.wikipedia.org/wiki/Maxwell%27s_equations)

Screenshot

Description

These links explain Gauss’ flux theorem and Maxwell’s equations.

Purpose

These links provide the learner with additional information on Gauss’ theorem, based on Maxwell’s equations.
Link No. 11

**URL title:** [http://fr.wikipedia.org/wiki/D%C3%A9riv%C3%A9](http://fr.wikipedia.org/wiki/D%C3%A9riv%C3%A9) [in French]


**Screenshot**

**Description**

This link provides an explanation of the derivative of a function: definition, notation, common derivative functions and the rules of derivation. It also provides an intuitive description and a more rigorous description of differentiation.

**Purpose**

This link provides the learner with more information on the derivatives of a function in order to solve some of the exercises in the module.
Link No. 12

URL title: [http://www.cpe.fr/gmm_cours/chimie_quantique/fb1997/mq2/mq2.htm][not found]

Screenshot

Description
This link describes some applications of integrals in the physical sciences.

Purpose
This link provides the learner with more information on applications of mathematics in the physical sciences, particularly integrals.

[No suggestions for English links]
African Virtual University

Link No. 13

URL title: http://perso.orange.fr/godzatswing/sommairemath.html [page not found]

[No suggestions for English link]

Screenshot

Description

This link covers some mathematical tools used in physics, particularly analytic, multiform, algebraic and transcendental functions.

Purpose

All the functions covered in this link are fundamental concepts used in this module. The learner should know the differences between them in order to solve some of the exercises in the Summary assessment.
Link No. 14

URL title: http://assocampus.ifrance.com/pages/rapso.htm
[in French]
http://navify.com/article/del

Description

These links cover most of the themes dealt with in the mandatory sections of the module: from scalar product to gradient product, including vector flux(or)flow and Stokes’ theorem.

Purpose

These links provide the learner with more information in order to solve most of the learning exercises in this module as well as the exercises in the Summary assessment.
XIII. Synthesis of the Module

This module provides the learner with information on the basic methods of vector analysis that are currently used to study physical phenomena. Learners should complete the entire module and make sure they fully understand the concepts. After doing the required reading, learners must complete the exercises in each learning activity through a self-learning process. The exercises provide practice in determining the coordinates of a vector and calculating a scalar product, a vector product, a mixed product and a double product of two or more vectors.

Calculation methods are explained for total and partial vector derivatives and for functions such as vector gradients and simple or multiple integrals of functions.

The notions of the vector field, scalar field, field lines, differential operators and flux are explained.

Upon completion of the module, the learner should have achieved the following objectives:

- Understand basic notions of vector algebra.
- Understand the tools of differential and integral calculations applied to functions (scalar and vector) of multiple variables.
- Use vector differentials.
- Understand the gradient and other vector analysis operators, particularly for geometric and physical interpretations.
- Understand and apply Stokes’ and Gauss’ theorems.
XIV. Summative Evaluation

Exercise 1

Given the vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) in a Cartesian system defined by:
\[
\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{i} + \hat{j} + \hat{k}, \quad \vec{c} = \hat{i} - 3\hat{j} + 2\hat{k},
\]
a) Do these three vectors constitute a three-dimensional base?
   A - Yes   B - No

b) If yes, which of the following are the components of the vector \( \vec{V} = 2\hat{i} - 4\hat{j} - \hat{k} \) in this system?
   A - \( \left(\frac{3}{2}, \frac{9}{2}, 4\right) \)   B - \( \left(\frac{9}{2}, -4, \frac{3}{2}\right) \)   C - \( (2, -4, -1) \)   D - \( (1, 1, 1) \)

Exercise 2

Given the vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) in a three-dimensional space,
a) If \( \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \), then does \( \vec{b} = \vec{c} \)?
   A - Yes   B - No

b) If \( \vec{a} \perp \vec{b} \), then do we have \( \vec{a} \wedge (\vec{a} \wedge \vec{b}) = -\vec{a}^2 \)?
   A - Yes   B - No

Exercise 3

Given the vector function \( \vec{V}(t) = e^{2t}(\cos 2t \hat{i} + \sin 2t \hat{j} + t \hat{k}) \),
calculate the first derivative \( \vec{V}'(t) \) and the second derivative \( \vec{V}''(t) \).

Exercise 4

Given the vector function \( \vec{r}(t) = r(t)\hat{u}(\theta) \) with \( \theta = \theta(t) \).
a) Express \( \frac{d\vec{r}}{dt} \) and \( \frac{d^2\vec{r}}{dt^2} \).
b) Calculate $\mathbf{C} = \mathbf{r} \times \frac{d\mathbf{r}}{dt}$, assuming that $\mathbf{r}$ and $\frac{d^2\mathbf{r}}{dt^2}$ are parallel and the end point $M$ of the vector $\overrightarrow{OM}(t) = \mathbf{r}(t)$ describes a continuous curve in the plane passing through $O$ and perpendicular to the vector $\mathbf{C}$.

Exercise 5

Calculate the integral vector: $\int_0^{\pi/2} (\cos 3t \mathbf{i} + \sin 2t \mathbf{j}) dt$.

Exercise 6

Given a material particle that describes a curve in space defined by the following parameterized equations (where $t$ is the time):

$$x = 2e^t \sin t$$

$$y = 2e^t \cos t$$

$$z = e^t$$

a) Determine the velocity and acceleration vectors of the particle at time $t$.

b) Determine the radius of the trajectory curve at point $t = 0$.

c) Determine the unit tangent vectors, the principal normal vector and the binormal vector at point $t = 0$.

Exercise 7

Given a surface defined by the parametric equation:

$$x = R \cos u \cos v$$

$$y = R \sin u \cos v$$

$$z = R \sin v$$

a) Determine the Cartesian equation for this surface.

b) Determine the unit vector normal to the surface.

c) Determine the equation of the plane tangent to point $M(u = 0, v = \frac{\pi}{4})$.

d) Once you have defined a surface element, calculate the total area of the surface.
You must define the appropriate limits for this calculation.

e) Calculate the flux over the vector field \( \frac{\mathbf{r}}{r^2} \).

**Exercise 8**

Are the following forces gradient fields? If yes, determine for each case the scalar potential from which the force field is derived.

a) \( \mathbf{F} = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz\mathbf{k} \)

b) \( \mathbf{F} = (3x^2z + y)\mathbf{i} + (z^2 + x)\mathbf{j} + (x^3 + 2yz)\mathbf{k} \)

**Exercise 9**

When a solid body rotates around a fixed axis \( D \), the velocity distribution of the points \( M \) of the solid is given by:

\[ \mathbf{v}(M) = \omega \times \mathbf{OM} \]

where \( \omega \) is a vector (constant) parallel to \( D \), called the instantaneous rotation velocity of the solid, and \( O \) is any point on \( D \). We impose \( OM = \mathbf{r} = xi + yj + z\mathbf{k} \). Calculate the rotational of the vector velocity field.

**Exercise 10**

An electromagnetic wave is composed of an electric field \( \mathbf{E} \) and a magnetic field \( \mathbf{B} \). In a vacuum, these fields are related by Maxwell’s equations, as follows:

\[
\begin{align*}
(1) \ & \text{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
(2) \ & \text{div} \mathbf{B} = 0 \\
(3) \ & \text{div} \mathbf{E} = 0 \\
(4) \ & \text{rot} \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

a) Which one is the propagation equation for the electric field?

b) Show that a solution to the propagation equation is in the form \( \mathbf{E} = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \) to which is associated the complex representation \( \mathbf{E} = E_0 \exp(i(\omega t - \mathbf{k} \cdot \mathbf{r})) \). Why is this complex representation useful?
Exercise 11

Given:

The velocity of light in a vacuum: \( c = 3.10^8 \text{ m/s} \)

Vacuum permittivity: \( \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m} \)

Vacuum permeability: \( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \)

We propose to define the structure of a plane wave in a vacuum in the absence of any change in the charge or current. This space is represented by a direct orthonormal axis system \( Oxyz \), and we denote by \( \hat{e}_x, \hat{e}_y \) and \( \hat{e}_z \) the unit vectors on the three axes.

1) According to Maxwell’s equations, the equations for the partial derivatives of the electric field \( \vec{E} \) and the magnetic field \( \vec{B} \) are deduced.

2) Next, we hypothesize that the wave is planar (PW), that is, the fields \( \vec{E} \) and \( \vec{B} \) depend on only a single spatial variable, in this case, the \( z \) coordinate (the partial derivatives with respect to \( x \) and \( y \) are null) and on time \( t \). Also consider \( f(z,t) \) as any component of \( \vec{E} \) or \( \vec{B} \). The solution to an equation satisfied by \( f(z,t) \) is the sum of the two terms. Show that they correspond to two progressive waves that propagate in opposite directions. We may use the variables \( u = z - ct \) and \( v = z + ct \). Calculate the speed \( c \) of propagation for these waves.

3) Consider a progressive planar wave (PPW) crossing the \( z \) axis, and assume that the coordinates of \( \vec{E} \) and \( \vec{B} \) do not have a constant term. Now suppose that \( \vec{E} \) varies according to the law: \( \vec{E} = E_0 \hat{e}_x \cos \left( \frac{\omega}{c} (z - ct) \right) \). Show that the fields \( \vec{E} \) and \( \vec{B} \) are transverse, are orthogonal, and that the axis system \( (\vec{E}, \vec{B}, \vec{c}) \) is direct (\( \vec{c} \) is a vector that can be defined).

Give the law of variation for \( \vec{B} \).

Determine the relationship between \( \varepsilon_0, \mu_0, E_0, c, B_0 \).
Numerical application: For \( \omega = 2\pi \cdot 10^{10} \text{ rad.s}^{-1} \), \( E_0 = 100 \text{ V.m}^{-1} \), calculate \( B_0 \) and the wave length \( \lambda \).

4) The impedance of a wave is known as the quantity \( Z_c = \frac{\mu_0 E}{B} \). Give the expression of \( Z_c \) as a function of the characteristics of vacuum as well as its numerical value.

5) Electromagnetic energy density is denoted by \( w(M,t) \), and the Poynting vector is denoted by \( \vec{P}(M,t) \) (i.e., energy current density, and is also known as energy flux).

Determine the local area that represents the conservation of electromagnetic energy.

Deduce from Maxwell’s equations the expressions of \( w \) and \( \vec{P} \) as a function of \( \vec{E} \) and \( \vec{B} \).

Give a simple relationship between \( \vec{P} \) and \( w \) for the case of a PPW as a function of the electric field \( \vec{E} \).

Exercise 12

A space is represented by an orthonormal axis system \( Ox, Oy \) and \( Oz \). A plane surface parallel to \( Oxy \) and equation \( z = 0 \) separates the space into two parts:

- The area \( z > 0 \) is empty;
- The area \( z < 0 \) is composed of a perfect dielectric medium (containing no charge, no current, no internal volume or surface irregularities), and the magnetic permeability is that of the vacuum \( \mu_0 \) and electric permittivity is \( \varepsilon \).

Consider a PPW with angular speed (pulsation) \( \omega \), polarized rectilinearly (electric field parallel to the \( Oy \) axis) and propagating through the dielectric medium (occupying the half-space \( z < 0 \)) defined along the wave vector \( \vec{k} \) which is located in the \( xOz \) plane and forming an angle \( \theta \) with the axis \( Ox \) (\( 0 < \theta < \frac{\pi}{2} \)). This
wave varies only with the coordinates $x$ and $z$. Therefore, it is the incidental wave, and it arrives at the separating surface between the two media, where it generates a reflected wave and a transmission wave. The incident wave is indicated by the index $i$ and its electric field has an amplitude $E_{0i}$ (real).

1) Express the speed $c_D$ of the wave in the dielectric as a function of $\varepsilon_0$ (electric permittivity of the vacuum) $\varepsilon$ and $c$ (velocity of light in the vacuum).

2) We denote by $\vec{k}_r$ the wave vector of the reflected wave and by $\vec{k}_0$ the wave vector of the transmission wave. The amplitudes of the electric field of the
   • reflected and transmission wave are respectively denoted by $E_{0r}$ and $E_{0t}$.
   • Provide the complex expressions of the electric and magnetic fields associated with the three waves.
   • Explain why the reflected and transmission waves have the same pulsation as the incident wave.
   • Determine the modulus of the wave vectors $\vec{k}$, $\vec{k}_r$ and $\vec{k}_0$.

3) What other wave-vector conditions define the passage through the fields? Using the Snell-Descartes Law,
   • explain the general form of the non-vanishing (non-null) components of the electric and magnetic fields as a function of $x$, $z$, the wave vectors $k$ and the angles $q$;
   • determine the Fresnel coefficients of reflection and transmission.

4) Show that for $0 < \theta < \theta_c$, where $\theta_c$ is a defining boundary value, total reflection occurs.
   • Does this phenomenon preclude a transmission wave? If this wave does indeed exist, what are its characteristics along $Ox$ and $Oz$?
   • For what value of $\delta_\varepsilon$ and $\varepsilon$ is the amplitude of the transmitted field divided by $e$?
   • Determine the average value of the Poynting vector. Comment on this result.
**Solution to exercise 1**

a) The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ constitute a three-dimensional base if they are linearly independent, in other words, $\lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c} = \vec{0} \iff \lambda_1 = \lambda_2 = \lambda_3 = 0$.

We can have this equality only if the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} \neq 0$$

which is the case when $D = 4$. Therefore, the vectors in question constitute a three-dimensional system.

**Answer is case A.**

b) We have: $\vec{V} = 2\vec{i} - 4\vec{j} - \vec{k} = x\vec{a} + y\vec{b} + z\vec{c}$.

Thus:

$$\vec{V} = x(\vec{i}) + y(\vec{j} + \vec{k}) + z(\vec{i} - 3\vec{j} + 2\vec{k})$$

$$= (x + y + z)\vec{i} + (x + y - 3z)\vec{j} + (y + 2z)\vec{k}$$

which leads to the following equation system:

$$\begin{align*}
x + y + z &= 2 \\
x + y - 3z &= -4 \\
y + 2z &= -1
\end{align*}$$

for which the solution is $x = \frac{9}{2}$, $y = -4$, $z = \frac{3}{2}$.

**Answer is case B.**
Solution to exercise 2

a) We have: \( \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \iff \vec{a} \cdot (\vec{b} - \vec{c}) = 0 \).

Therefore, if \( \vec{b} \neq \vec{c} \), then \( \vec{a} \perp (\vec{b} - \vec{c}) \). Finally, \( \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \) does not obtain \( \vec{b} = \vec{c} \).

Answer is case B

b) We know that \( \vec{a} \wedge (\vec{a} \wedge \vec{b}) = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} \).

Thus, if \( \vec{a} \perp \vec{b} \), then \( \vec{a} \cdot \vec{b} = 0 \), whence \( \vec{a} \wedge (\vec{a} \wedge \vec{b}) = -\vec{a} \cdot \vec{b} \).

Answer is case A

Solution to exercise 3

We have:

\[
\frac{d\vec{V}}{dt} = \vec{V}'(t) = 2e^{2t}(\cos 2t - \sin 2t)\hat{i} + 2e^{2t}(\sin 2t + \cos 2t)\hat{j} + 2e^{2t}(2t + 1)\hat{k}
\]

and

\[
\frac{d^2\vec{V}}{dt^2} = \vec{V}''(t) = 8e^{2t} \left[ (-\sin 2t)\hat{i} + (\cos 2t)\hat{j} + (t + 1)\hat{k} \right]
\]

Solution to exercise 4

a) \( \vec{r}(t) = r(t)\hat{u} \iff \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{u} + r \frac{d\hat{u}}{dt} \frac{d\theta}{dt} \)

and

\[
\frac{d^2\vec{r}}{dt^2} = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right]\hat{u} + \left[ 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right] \frac{d\hat{u}}{dt} \frac{d\theta}{dt}
\]
b) The vector $\vec{C} = \vec{r} \wedge \frac{d\vec{r}}{dt}$ is given by:

$$\vec{C} = \vec{r} \wedge \left( \frac{d\vec{r}}{dt} + \vec{u} \frac{du}{dt} \right) = \vec{r} \wedge \frac{du}{dt}$$

If $\vec{r} \parallel \frac{d^2\vec{r}}{dt^2}$, we have:

$$\frac{d\vec{C}}{dt} = \frac{d}{dt} (\vec{r} \wedge \frac{d\vec{r}}{dt}) = \frac{d\vec{r}}{dt} \wedge \frac{d\vec{r}}{dt} + \vec{r} \wedge \frac{d^2\vec{r}}{dt^2} = 0$$

The vector $\vec{C} = \vec{r} \wedge \frac{d\vec{r}}{dt}$ is therefore a constant vector.

- If $\vec{C} = 0$, then $\frac{du}{dt} = 0$, the vector $\vec{u}$ is constant and the point $M$ describes the right-hand $(O;\vec{u})$.

- If $\vec{C} \neq 0$, $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are always perpendicular to $\vec{C}$, which is a constant vector, in other words having a fixed direction and constant modulus, the point $M$ describes a curve located in the plane passing through $O$ and orthogonal to vector $\vec{C}$.

**Solution to exercise 5**

$$\int_0^{\pi/2} (\cos 3t i + \sin 2t j) dt = \left[ \frac{1}{3} \sin 3t \right]_0^\pi + \left[ -\frac{1}{2} \cos 2t \right]_0^\pi = \frac{\pi}{3} i - \frac{1}{2} j$$

**Solution to exercise 6**

a) The velocity vector is given by:

$$\vec{v} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

Thus:
\[
\begin{align*}
\frac{dx}{dt} &= 2e^t (\sin t + \cos t) \\
\frac{dy}{dt} &= 2e^t (\cos t - \sin t) \\
\frac{dz}{dt} &= e^t
\end{align*}
\]

For the acceleration vector, we have:

\[
a = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} + \frac{d^2z}{dt^2} \hat{k}
\]

Thus:

\[
\begin{align*}
\frac{d^2x}{dt^2} &= 4e^t \cos t \\
\frac{d^2y}{dt^2} &= -4e^t \sin t \\
\frac{d^2z}{dt^2} &= e^t
\end{align*}
\]

b) From the previous question, the square of the velocity vector modulus is given by:

\[
v^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 = 9e^{2t}, \text{ thus } v = 3e^t.
\]

The tangential acceleration is therefore:

\[
a_t = \frac{dv}{dt} = 3e^t.
In addition, we have:

\[ a^2 = a_t^2 + a_n^2 = 17e^{2t}, \] which gives:

\[ a_n = \sqrt{8e^t} = \frac{v^2}{R_c} \]

Thus

\[ R_c = \frac{v^2}{a_n} = \frac{9}{2\sqrt{2}} e^t, \] and for \( t = 0 \), we have \( R_c = \frac{9}{2\sqrt{2}} \)

c) The unit tangent vector is given by:

\[ \vec{T} = \frac{\vec{v}}{v} = \frac{2}{3} (\sin t + \cos t) \hat{i} + \frac{2}{3} (\cos t - \sin t) \hat{j} + \frac{1}{3} \hat{k} \]

Thus, for \( t = 0 \)

\[ \vec{T}_0 = \frac{2}{3} \hat{i} + \frac{2}{3} \hat{j} + \frac{1}{3} \hat{k} \]

The unit normal vector is such that:

\[ \frac{d\vec{T}}{ds} = \frac{1}{R_c} \vec{N} \text{ or } \frac{d\vec{T}}{ds} / \frac{dt}{dt} = \frac{2}{9} e^{-t} (\cos t - \sin t) \hat{i} - \frac{2}{9} e^{-t} (\sin t + \cos t) \hat{j} \]

and

\[ \left\| \frac{d\vec{T}}{ds} \right\| = \sqrt{\frac{8}{9} e^{-2t}} = \frac{1}{R_c}, \] and we can determine the value of the curve radius.

Thus,

\[ \vec{N} = \frac{1}{\sqrt{2}} (\cos t - \sin t) \hat{i} - \frac{1}{\sqrt{2}} (\sin t + \cos t) \hat{j} \]
The unit binormal vector is given by: $B = T \wedge N$, thus

$$B_0 = \frac{1}{3\sqrt{2}} i + \frac{1}{3\sqrt{2}} j - \frac{4}{3\sqrt{2}} k$$

Solution to exercise 7

a) The Cartesian equation for the trajectory is obtained by eliminating the parameters $u$ and $v$, as follows:

$$x^2 + y^2 + z^2 = R^2$$

This is the equation for a sphere having centre $O$ and radius $R$.

b) A normal vector of the surface at point $M$ is given by:

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \wedge \frac{\partial \vec{r}}{\partial v}$$

where $\vec{r} = OM$

$$\frac{\partial \vec{r}}{\partial u} = -R \sin u \cos v \hat{i} + R \cos u \cos v \hat{j}$$

and

$$\frac{\partial \vec{r}}{\partial v} = -R \cos u \sin v \hat{i} - R \sin u \sin v \hat{j} + R \cos v \hat{k}$$

Thus, $\vec{N} = R^2 \cos u \cos^2 v \hat{i} + R^2 \sin u \cos^2 v \hat{j} + R^2 \cos v \sin v \hat{k}$, and therefore the unit normal vector is given by:

$$\hat{n} = \frac{\vec{N}}{||\vec{N}||} = \cos u \cos v \hat{i} + \sin u \cos v \hat{j} + \sin v \hat{k}$$
c) If we denote by $\mathbf{r}_o$ the position of point $M (u = 0, v = \frac{\pi}{4})$, and by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the position of any point on the tangent plane, then we have: $(\mathbf{r} - \mathbf{r}_o) \cdot \mathbf{n}_0 = 0$ where $\mathbf{n}_0$ is the unit normal vector to the surface at $M$.

Thus:

$$(\mathbf{r} - \mathbf{r}_o) \cdot \mathbf{n}_0 = \left[ (x - \frac{\sqrt{2}}{2} R)\mathbf{i} + y\mathbf{j} + (z - \frac{\sqrt{2}}{2} R)\mathbf{k} \right] \cdot \left[ \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{k} \right] = 0$$

Whence we derive the equation for the tangent plane: $x + z = \sqrt{2}R$

d) The area? (or) surface? element at a point on the surface is given by:

$$dS = \left\| \mathbf{N} \right\| du dv = R^2 \cos v du dv$$

Thus, the total area? (or) surface? of the sphere is given by:

$$S = R^2 \int_0^{2\pi} du \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos v dv = 4\pi R^2$$

e) The constant field modulus on the sphere is given by: $\frac{1}{R^2}$, therefore the field flux is given by:

$$\Phi = \iint_{(\text{sphère})} \frac{r}{r^3} \cdot d\mathbf{S} = \frac{1}{R^2} \iiint_{(\text{sphère})} dS = 4\pi$$
Solution to exercise 8

For a force field to be a gradient field, it is sufficient that it has null rotation, as follows: \( \text{rot} \vec{F} = \vec{0} \).

a) 
\[
\text{rot} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2y & xy^2 & xz^2
\end{vmatrix} = (xz^2 - xy^2)\hat{i} - (yz^2 - x^2y)\hat{j} + (y^2z - x^2z)\hat{k}
\]

This force field is not a gradient field. It therefore does not derive from a scalar potential.

b) 
\[
\text{rot} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3x^2z + y & z^3 + x & x^3 + 2yz
\end{vmatrix} = 0
\]

This field is a gradient field. We therefore have:

\[
\vec{F} = \text{grad} U = \frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} + \frac{\partial U}{\partial z} \hat{k}
\]

Thus:

\[
\frac{\partial U}{\partial x} = 3x^2 + y \Rightarrow U(x, y, z) = x^3z + xy + f_1(y, z)
\]

\[
\frac{\partial U}{\partial y} = z^3 + x \Rightarrow U(x, y, z) = yz^2 + xy + f_2(x, z)
\]

\[
\frac{\partial U}{\partial z} = x^3 + 2yz \Rightarrow U(x, y, z) = x^3z + yz^2 + f_3(x, y)
\]

The equality of these three expressions leads to:

\[U(x, y, z) = x^3z + xy + yz^2 + cte\]
Solution to exercise 9

The velocity vector is given by:

\[
\vec{v}(M) = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (z\omega_y - y\omega_z)\vec{i} + (x\omega_z - z\omega_x)\vec{j} + (y\omega_x - x\omega_y)\vec{k}
\]

The rotational vector velocity is:

\[
\text{rotv} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\omega_y - y\omega_z & x\omega_z - z\omega_x & y\omega_x - x\omega_y \end{vmatrix} = 2(\omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}) = 2\vec{\omega}
\]

Note that the rotational of the vector velocity field of a rotating solid is equal to twice the instantaneous rotational velocity of the vector, hence the name of this operator.

Solution to exercise 10

a) We may calculate the rotational of equation (1) as follows:

\[
\nabla \times (\nabla \times \vec{E}) = \nabla \times (-\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}
\]

Furthermore, we have:

\[
\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}
\]

We therefore obtain the equation for the electric field propagation, as follows:

\[
\nabla^2 \vec{E} - \varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2} = 0
\]

By analogy, if we eliminate the electric field, we obtain the equation for the magnetic field propagation:

\[
\nabla^2 \vec{B} - \varepsilon_0\mu_0\frac{\partial^2 \vec{B}}{\partial t^2} = 0
\]
b) This complex representation is useful in analytical operations such as the derivation and integration of algebraic multiplication and division. In this case, the derivative operator \( \vec{\nabla} \) expressed in Cartesian coordinates is particularly simple. We have:

\[
\vec{\nabla} \leftrightarrow -i\vec{k} \quad \text{and} \quad \frac{\partial}{\partial t} \leftrightarrow i\omega
\]

which gives:

\[
\vec{\nabla}^2 \vec{E} = -k^2 \vec{E} \quad \text{and} \quad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}
\]

We therefore obtain:

\[
(\omega^2\epsilon_0\mu_0 - k^2)\vec{E} = 0 \quad \text{where} \quad \omega = c|k|, \quad \text{and} \quad c = \frac{1}{\sqrt{\epsilon_0\mu_0}} \quad \text{is the celerity of light in a vacuum.}
\]

Maxwell’s equations become:

\[
\begin{align*}
(1) \quad -i\vec{k} \wedge \vec{E} &= -i\omega \vec{B} \\
(2) \quad -i\vec{k} \cdot \vec{E} &= 0 \\
(3) \quad -i\vec{k} \cdot \vec{B} &= 0 \\
(4) \quad -i\vec{k} \wedge \vec{B} &= i\omega\epsilon_0\mu_0 \vec{E}
\end{align*}
\]

The three vectors \((\vec{E},\vec{B},\vec{k})\) therefore form a direct orthogonal system with

\[
B = \frac{E}{c}
\]
Solution to exercise 11

1) Maxwell’s equations in a vacuum, in the absence of either charge or current, are written as:

(1) $\nabla \cdot \vec{E} = 0$ \hspace{1cm} (2) $\nabla \cdot \vec{B} = 0$

(3) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ \hspace{1cm} (4) $\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

To obtain the equations for the partial derivatives, we can eliminate, for example, $\vec{B}$ from the two last relationships, by taking the rotational of (3) and using the value obtained from (4), which gives:

$\nabla \times \nabla \cdot \vec{E} = \nabla \cdot \nabla \times \vec{E} = -\Delta \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{B} = -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

whence we get the wave equation:

$\Delta \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

In the same way, eliminating $\vec{E}$, we obtain:

$\Delta \vec{B} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$

The Laplacian is written in Cartesian coordinates as follows:

$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

By applying this operator to each component of the electric vector field (for example), we obtain three equations, as follows:

$\frac{\partial^2 \vec{E}_i}{\partial x^2} + \frac{\partial^2 \vec{E}_i}{\partial y^2} + \frac{\partial^2 \vec{E}_i}{\partial z^2} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}_i}{\partial t^2}$ with $i = x, y, z$
By solving this wave equation for the partial derivatives, we get the expression of the $E$ and $B$ fields as a function of the spatial coordinates and time $t$.

2) Now we can solve this equation for the case of a plane wave, in which the fields vary only in the direction of propagation (here, the $Oz$ axis), which implies that all the partial derivatives are null in the $Oxy$ plane, as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

Thus, each of the six $E_i$ and $B_i$ components confirms a wave equation in one dimension in the form:

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \quad \text{with} \quad c^2 = \frac{1}{\varepsilon_0 \mu_0}.$$ 

By introducing the variables $u = z - ct$ and $v = z + ct$, we then obtain:

$$z = \frac{1}{2} (u + v) \quad \text{and} \quad t = \frac{1}{2c} (u - v).$$

Thus, $f(z,t) = f(u,v)$.

We then have:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}.$$ 

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \right) = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} + 2 \frac{\partial^2 f}{\partial u \partial v}.$$ 

In the same way, we have:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial t} = c \left( -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \right).$$

and
\[
\frac{\partial^2 f}{\partial t^2} = c^2 \left( \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} - 2 \frac{\partial^2 f}{\partial u \partial v} \right)
\]

Therefore, the wave equation in the form \( \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \) is reduced to:

\[
\frac{\partial^2 f}{\partial u \partial v} = 0
\]

and:

\[
\frac{\partial}{\partial u} \left( \frac{\partial f}{\partial v} \right) = 0
\]

which, by integration, gives:

\[
\frac{\partial f}{\partial v} = g(v), \text{ the function of the single variable } v, \text{ whose integration leads to:}
\]

\[
f(u, v) = F(u) + G(v)
\]

where \( G(v) \) is a primitive of \( g(v) \) and \( F(u) \) is the integration constant related to \( v \). The general solution of the wave equation \( f(z, t) \) is written as:

\[
f(z, t) = F(z - ct) + G(z + ct)
\]

where \( F \) and \( G \) are any functions. This progressive plane wave equation moves in the direction \( Oz \), with:

- \( F(z - ct) \) representing the direct progressive wave propagating with speed \( c \) in the direction of the \( z \) crossings. If at time \( t \) and at abscissa point \( z \) the function has the value \( F(z - ct) \), it will also have exactly the same value at time \( t + \Delta t \) at the abscissa point \( z + \Delta z \). And at the end of time interval \( \Delta t \), the field appears identical to itself as it travels along the \( z \) axis for a distance \( \Delta z = c \Delta t \), so we can say that it propagates at constant speed \( c \). The constant \( c \) therefore stands for the propagation rate or field speed. The wave is described as a plane wave because the wave surfaces \( (z = \text{constant at a fixed time}) \) are planes that are orthogonal to the direction of propagation, with the same value at every point on the plane wave.
• $G(z + ct)$ represents the retrograde wave that propagates with velocity $c$ along the $Oz$ axis, but in the opposite direction.

The behaviours of the quantities associated with a magnetic field are regulated by the relationships that describe the propagation in a vacuum with a velocity (speed of light in a vacuum), as follows:

\[
c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \text{ or approximately } c = 3.10^8 \text{ m/s}.
\]

3) Maxwell’s equations (1) and (2) lead to:

\[
\begin{align*}
\frac{\partial E_z}{\partial z} &= 0 \text{ et } \frac{\partial B_z}{\partial z} = 0, \text{ whence the } \vec{E} \text{ and } \vec{B} \text{ fields do not have components along the } z \text{ axis: the fields are therefore transverse, and the wave is a transverse electromagnetic wave (TEM).}
\end{align*}
\]

• Maxwell’s third equation gives:

\[
\begin{align*}
\frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t} = 0 \\
\frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} = -\frac{\omega}{c} E_0 \sin \left[ \frac{\omega}{c} (z - ct) \right] \\
0 &= -\frac{\partial B_z}{\partial t}.
\end{align*}
\]

We then have: $B_y = \frac{E_x}{c}$, whence $\vec{E} \cdot \vec{B} = 0$, and $\|\vec{E}\| = c \|\vec{B}\|$. The $\vec{E}$ and $\vec{B}$ fields are therefore orthogonal, and proportional to the standard. As a result: $\vec{E} = \vec{B} \wedge c \vec{e}_z = \vec{B} \wedge \vec{c}$, noting that $\vec{c} = c \vec{e}_z$ is the wave velocity vector. The reference frame $(\vec{E}, \vec{B}, \vec{c})$ is therefore direct.

• The law of variation in $\vec{B}$ is given by:

\[
\vec{B} = \frac{E_0}{c} \vec{e}_y \cos \left[ \frac{\omega}{c} (z - ct) \right].
\]
We have: \( B_0 = \frac{E_0}{c} = \sqrt{\varepsilon_0 \mu_0} E_0 \)

- Numerical application:

\[
B_0 = \frac{100 \text{V} \text{ / m}}{3.10^8 \text{m} / \text{s}} \approx 3.10^{-7} \text{T}
\]

\[
k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \Rightarrow \quad \lambda = \frac{2\pi c}{\omega} = 3.10^{-2} \text{m}
\]

4) The characteristic impedance of a vacuum is given by:

\[
Z_c = \frac{\mu_0}{\varepsilon_0} = \mu_0 c = \mu_0 \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{\mu_0}{\varepsilon_0}
\]

A.N. \( Z_c = 377 \Omega \)

5) The principle of local conservation of energy states that the rate of temporal variation in the electromagnetic field energy enclosed in a fixed volume in space is equal to the radiation intensity of the field propagation increased by the electromagnetic intensity received by the charge carriers contained in the volume. However, theoretically, there is no charge in a vacuum. We therefore have:

\[
- \frac{dE_{\text{em}}}{dt} = - \frac{d}{dt} \iiint_{\text{vol}} \omega(M, t) \, dt = \iint_S \vec{P} \cdot d\vec{S}
\]

Thus, by applying the Ostrogradski theorem:

\[
- \frac{d}{dt} \iiint_{\text{vol}} \omega(M, t) \, dt = \iint_S \vec{P} \cdot d\vec{S} = \iiint_{\text{vol}} \text{div} \, \vec{P} \, d\tau
\]
Finally, it becomes:

\[
\frac{\partial w}{\partial t} + \text{div } P = 0
\]

- Using Maxwell’s equations (3) and (4), multiplied scalarly by \( \bar{B} \) and \( \bar{E} \) respectively, we obtain:

\[
\bar{B} \cdot \text{rot } \bar{E} = \bar{E} \cdot \text{rot } \bar{B} = -\bar{B} \cdot \frac{\partial \bar{B}}{\partial t} = -\frac{1}{2} \frac{\partial \bar{B}^2}{\partial t},
\]

\[
\bar{E} \cdot \text{rot } \bar{B} = \varepsilon_0 \mu_0 \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} = \frac{1}{2} \varepsilon_0 \mu_0 \frac{\partial \bar{E}^2}{\partial t}.
\]

By adding these equations member to member, we have:

\[
\bar{E} \cdot \text{rot } \bar{B} - \bar{B} \cdot \text{rot } \bar{E} = -\text{div}(\bar{E} \wedge \bar{B}) = \frac{1}{2} \varepsilon_0 \mu_0 \frac{\partial \bar{E}^2}{\partial t} + \frac{1}{2} \frac{\partial \bar{B}^2}{\partial t}.
\]

and:

\[
\text{div}(\frac{\bar{E} \wedge \bar{B}}{\mu_0}) + \frac{\partial}{\partial t}(\frac{\varepsilon_0 \bar{E}^2}{2} + \frac{\bar{B}^2}{2\mu_0}) = 0.
\]

We can identify this equation with the equation of local energy conservation by imposing:

\[
w = \frac{\varepsilon_0 \bar{E}^2}{2} + \frac{\bar{B}^2}{2\mu_0} \quad \text{and} \quad P = \frac{\bar{E} \wedge \bar{B}}{\mu_0}.
\]

Note that the density of energy \( w \) is defined at a constant additive speed set at null to prevent energy in the absence of an electric field. Similarly, the Poynting vector is defined at a rotational speed so that \( P \) is null in the absence of a field and parallel to the direction of propagation.
For a PPW, we have:

\[
\vec{E} \wedge \vec{B} = \vec{E} \wedge \left( \frac{\vec{e}_z \wedge \vec{E}}{c} \right) = \frac{\vec{E}^2}{c} \vec{e}_z
\]

Which gives, for the Poynting vector:

\[
\vec{P} = \frac{\vec{E}^2}{c \mu_0} \vec{e}_z = c \varepsilon_0 \vec{E}^2 \vec{e}_z = \varepsilon_0 \vec{E}^2 \frac{c}{c}
\]

The energy density is therefore:

\[
\omega = \frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2 \mu_0} = \frac{\varepsilon_0 \vec{E}^2}{2} + \frac{1}{2 \mu_0} \frac{\vec{E}^2}{c^2} = \varepsilon_0 \vec{E}^2
\]

And the Poynting vector is written as:

\[
\vec{P} = \omega \vec{c}
\]

Thus, in a vacuum, the propagation speed of electromagnetic energy is equal to the speed \( c \) of light.

Solution to exercise 12

1) The celerity of the electromagnetic wave through a medium is given by:

\[
\omega = \frac{1}{k} = \frac{1}{\sqrt{\varepsilon \mu_0}} = c \sqrt{\frac{\varepsilon_0}{\varepsilon}} = \frac{c}{n}
\]

where \( n = \sqrt{\varepsilon_r} \) is the index of the dielectric medium.
2) The electric and magnetic fields of the three waves are (considering unbounded media):

For the incident wave:

\[ \mathbf{E}_i = \mathbf{E}_{0i} \exp(i(\omega t - \mathbf{k} \cdot \mathbf{r})) \quad \text{et} \quad \mathbf{B}_i = \mathbf{B}_{0i} \exp(i(\omega t - \mathbf{k} \cdot \mathbf{r})) \]

with \( \mathbf{B}_{0i} = \frac{1}{\omega} \mathbf{k} \wedge \mathbf{E}_{0i} \).

For the reflected wave:

\[ \mathbf{E}_r = \mathbf{E}_{0r} \exp(i(\omega_r t - \mathbf{k}_r \cdot \mathbf{r})) \quad \text{et} \quad \mathbf{B}_r = \mathbf{B}_{0r} \exp(i(\omega_r t - \mathbf{k}_r \cdot \mathbf{r})) \]

with \( \mathbf{B}_{0r} = \frac{1}{\omega} \mathbf{k} \wedge \mathbf{E}_{0r} \).

For the transmission wave:

\[ \mathbf{E}_t = \mathbf{E}_{0t} \exp(i(\omega_t t - \mathbf{k}_0 \cdot \mathbf{r})) \quad \text{et} \quad \mathbf{B}_t = \mathbf{B}_{0t} \exp(i(\omega_t t - \mathbf{k}_0 \cdot \mathbf{r})) \]

with \( \mathbf{B}_{0t} = \frac{1}{\omega} \mathbf{k} \wedge \mathbf{E}_{0t} \).

- The two media are magnetically identical, whence the conservation of the magnetic field at plane \( z = 0 \), thus:

\[ \mathbf{B}_1 = \mathbf{B}_2 \Rightarrow \mathbf{B}_i + \mathbf{B}_r = \mathbf{B}_t \]

This relationship must be verified at each instant and for every point on the surface of separation, which implies the same temporal dependence for the three fields, and therefore:

\[ \omega = \omega_r = \omega_t. \]

- By definition, we have:

\[ k = \left\| \mathbf{k} \right\| = \frac{\omega}{c}, \text{ therefore } \mathbf{k} = \mathbf{k}_r = \mathbf{k}_t = \frac{\omega}{c} = \frac{2\pi n}{\lambda_0} = \frac{2\pi}{\lambda_0} \text{ and } \mathbf{k}_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}. \]
3) Given:

∀ \mathbf{r} \in \text{at the plane of separation of the two media, we have:}

\[ k \mathbf{r} \cdot \mathbf{r} = k_t \mathbf{u}_t \cdot \mathbf{r} \Rightarrow k \mathbf{r} \cdot \mathbf{u}_t = k_t \mathbf{u}_t \cdot \mathbf{r} = k_t \mathbf{u}_t. \]

where \( \mathbf{u}_t \) is a unit vector tangent to the \( Oxy \) plane. During reflection or refraction, there is conservation of the tangential component of the wave vector. This relationship is particularly true for the unit vector \( \mathbf{e}_y \):

\[ k \mathbf{r} \cdot \mathbf{e}_y = k_t \mathbf{u}_t \cdot \mathbf{e}_y = k_t \mathbf{e}_y = 0. \]

We deduce from this that the wave vectors of the reflected and refracted waves are perpendicular to \( \mathbf{e}_y \). These wave vectors are therefore in the \( xOz \) plane, and, according to the first Snell-Descartes law: the reflected and refracted waves are in the incident plane.

Along \( \mathbf{e}_x \), we have:

\[ k \mathbf{e}_x \cdot \mathbf{e}_x = k_t \mathbf{u}_t \cdot \mathbf{e}_x = k_t \mathbf{e}_x \Rightarrow k \cos \theta = k_0 \cos \theta' = k_0 \cos \theta_t, \]

and from this equality we derive:

\[ \theta = \theta', \text{ and } k \cos \theta = k_0 \cos \theta_t. \]

This translates (or) represents the second Snell-Descartes law: the incident angle is equal to the angle of reflection, and the angle of incidence and the angle of reflection are connected by:

\[ \frac{\sin i}{\sin r} = \frac{n_1}{n_2}, \]

with, in this case, \( i = \frac{\pi}{2} - \theta \) and \( r = \frac{\pi}{2} - \theta_t. \)

* The components of the different fields are expressed as

\[ \text{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \]
the incident wave:

\[ E_{ry} = E_{0i} \exp(i(\omega t - k_0x \cos \theta - k_0z \sin \theta)) \]

\[ B_{ix} = -\frac{E_{0i}}{c} \sin \theta \exp(i(\omega t - k_0x \cos \theta - k_0z \sin \theta)) \]

\[ B_{iz} = \frac{E_{0i}}{c} \cos \theta \exp(i(\omega t - k_0x \cos \theta - k_0z \sin \theta)) \]

the reflected wave:

\[ E_{ry} = E_{0r} \exp(i(\omega t - k_0x \cos \theta + k_0z \sin \theta)) \]

\[ B_{rx} = \frac{E_{0r}}{c} \sin \theta \exp(i(\omega t - k_0x \cos \theta + k_0z \sin \theta)) \]

\[ B_{rz} = \frac{E_{0r}}{c} \cos \theta \exp(i(\omega t - k_0x \cos \theta + k_0z \sin \theta)) \]

the transmission wave:

\[ E_{yt} = E_{0t} \exp(i(\omega t - k_0x \cos \theta_t - k_0z \sin \theta_t)) \]

\[ B_{tx} = -\frac{E_{0i}}{c} \sin \theta_t \exp(i(\omega t - k_0x \cos \theta_t - k_0z \sin \theta_t)) \]

\[ B_{tz} = \frac{E_{0i}}{c} \cos \theta_t \exp(i(\omega t - k_0x \cos \theta_t - k_0z \sin \theta_t)) \]

- The conditions for continuity at \( z = 0 \) impose that:
  - for the electric field (conservation of the tangential component):
    \[ E_{0i} + E_{0r} = E_{0t} \]
  - for the magnetic field: \( \overrightarrow{B}_{0i} + \overrightarrow{B}_{0r} = \overrightarrow{B}_{0t} \), thus:
\[ -\frac{E_{0i}}{c_q} \sin \theta + \frac{E_{0r}}{c_q} \sin \theta = -\frac{E_{0t}}{c} \sin \theta_t \]

and

\[ \frac{E_{0i}}{c_q} \cos \theta + \frac{E_{0r}}{c_q} \cos \theta = \frac{E_{0t}}{c} \cos \theta_t \]

Solving this equation system leads to:

\[ \frac{E_{0r}}{E_{0i}} = r = \frac{n \sin \theta - \sin \theta_t}{\sin \theta + \sin \theta_t} = \frac{\tan \theta - \tan \theta_t}{\tan \theta + \tan \theta_t} \]

and

\[ \frac{E_{0t}}{E_{0i}} = t = \frac{2n \sin \theta}{n \sin \theta + \sin \theta_t} = \frac{2 \tan \theta}{\tan \theta + \tan \theta_t} \]

4)

The Snell-Descartes law of refractions is written as follows:

\[ k \cos \theta = k_0 \cos \theta_c , \text{ so that we must have } \cos \theta_t \leq 1 , \text{ which is impossible if } \]

\[ \theta \in \left[ \theta_c, \frac{\pi}{2} \right] \text{ with } \cos \theta_c = \frac{k_0}{k} = \frac{1}{n} . \text{ Therefore, for } 0 < \theta < \theta_c , \text{ total reflection occurs, which does not mean that no wave is transmitted through the vacuum (medium 2), as this would be inconsistent with the passage relationships.} \]

This wave has a particular structure, as we shall see. For \( 0 < \theta < \theta_c \), we get

\[ \cos \theta_t = k - \cos \theta > 1 , \text{ with no sine wave, and it is therefore complex. Formally speaking, we could continue to apply the Snell-Descartes law by writing:} \]
\[
\sin \theta_t = \pm \sqrt{1 - \cos^2 \theta_t} = \pm \sqrt{1 - \frac{k^2}{k_0^2} \cos^2 \theta} = \pm \frac{q}{k_0} \text{ with } q = \sqrt{k^2 \cos^2 \theta - k_0^2}.
\]

We then obtain, for the transmitted electric field:

\[
E_{y_t} = E_{0t} \exp(i(\omega t - k_0 x \cos \theta_t - k_y z \sin \theta_t)) = E_{0t} \exp[i(\omega t - k x \cos \theta)]
\]

To prevent the field amplitude from extending exponentially to infinity, we choose

\[
\sin \theta_t = -i \frac{q}{k_0}, \text{ which gives:}
\]

\[
E_{y_t} = E_{0t} \exp[i(\omega t - k x \cos \theta)].
\]

This wave does not propagate in the direction of the \(Oz\) axis, where its amplitude decreases exponentially with distance. It is characteristic of an **evanescent wave**, which propagates parallel to \(Ox\), travelling along the positive \(x\) axis at a phase speed \(v_p = \frac{\omega}{k \cos \theta} = \frac{c}{n \cos \theta}\) imposed by the dielectric medium (medium 1) and the incident wave.

Note that the wave amplitude decreases perpendicularly to the direction of propagation. We therefore have a non-homogenous monochromatic plane progressive wave (MPPW), because its amplitude varies across the points of the wave plane.

The reflection coefficient can then be written as:

\[
r = \frac{1 - im}{1 + im} \text{ with } \frac{m}{i} = \frac{q}{k \sin \theta}.
\]

and therefore the amplitude of the reflected wave is equal to the amplitude of the incident wave.

- The transmission wave decreases exponentially in the direction perpendicular to the interface, such that its amplitude is reduced by a factor \(e\) for a distance \(d_z\) given by:

\[
d_z = \frac{1}{q} \frac{1}{\sqrt{k^2 \cos^2 \theta - k_0^2}} = \frac{\lambda_0}{2\pi \sqrt{n^2 \cos^2 \theta - 1}}.
\]
• From \( \text{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -i\omega \mathbf{B} \), we immediately get:

\[
\mathbf{B}_x = \frac{iq}{\omega} E_{0x} e^{zq} \exp(i(\omega t - kx \cos \theta))
\]

\[
\mathbf{B}_z = -\frac{k}{\omega} \cos \theta E_{0z} e^{zq} \exp(i(\omega t - kx \cos \theta))
\]

Unlike a homogenous MPPW, this magnetic field is not perpendicular to the phase propagation direction. The component along \( O_z \) is in phase with that of \( E_t \) whereas the component along \( O_x \) is in quadrature.

The average value of the Poynting vector is given by:

\[
\langle P_t \rangle = \frac{1}{2\mu_0} \text{Re}(\mathbf{E}_t \cdot \mathbf{B}_t^*)
\]

\[
\langle P_x \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu_0}} E_{0t}^2 e^{-2zq} e_z
\]

Therefore, on average, the energy of the evanescent wave that propagates parallel to the interface and the average force that crosses the surface of separation is null. This explains why all the energy of the incident wave is reflected, despite the presence of the transmission wave, whence the term total reflection. Medium 2 can be assimilated by pure induction, so the average force is null (nevertheless, the instantaneous value of the energy flux oscillates sinusoidally between two opposed values). However, there is still a reactive current of the inductance. In reality, the incident wave does not depart abruptly from the incident point, but instead penetrates into medium 2, from where it is sent back to medium 1. During this coming-and-going phase on the one hand and the separating from the surface phase on the other, the energy of the evanescent wave travels parallel to the interface.
XV. References


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MATHEMATICAL PHYSICS 2

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Del

In vector calculus, del is a vector differential operator, usually represented by the nabla symbol \( \nabla \). When applied to a function defined on a one-dimensional domain, it denotes its standard derivative as defined in calculus. When applied to a field (a function defined on a multi-dimensional domain), del may denote the gradient (slope) of a scalar field, the divergence of a vector field, or the curl (rotational) of a vector field, depending on the way it is applied.

Strictly speaking, del is not a specific operator, but rather a convenient mathematical notation for those three operators, that makes many equations easier to write and remember. The del symbol can be interpreted as a vector of partial derivative operators, and its three possible meanings—gradient, divergence, and curl—can be formally viewed as the scalar product, dot product, and cross product, respectively, of the del "operator" with the field. These formal products may not commute with other operators or products.

[] Definition

In the three-dimensional Cartesian coordinate system \( \mathbb{R}^3 \) with coordinates \((x, y, z)\), del is defined in terms of partial derivative operators as

\[
\nabla = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}
\]

where \(\{\hat{x}, \hat{y}, \hat{z}\}\) are the unit vectors in the respective coordinate directions.

Though this page chiefly treats del in three dimensions, this definition can be generalized to the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). In the Cartesian coordinate system with coordinates \((x_1, x_2, \ldots, x_n)\), del is:

\[
\nabla = \sum_{i=1}^{n} \hat{e}_i \frac{\partial}{\partial x_i}
\]

where \(\{\hat{e}_i : 1 \leq i \leq n\}\) is the standard basis in this space.

More compactly, using the Einstein summation notation, del is written as

\[
\nabla = \hat{e}_i \partial_i
\]

Del can also be expressed in other coordinate systems, see for example del in cylindrical and spherical coordinates.

[] Notational uses
Del is used as a shorthand form to simplify many long mathematical expressions. It is most commonly used to simplify expressions for the gradient, divergence, curl, directional derivative, and Laplacian.

[] Gradient

The vector derivative of a scalar field \( f \) is called the gradient, and it can be represented as:

\[
\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}
\]

It always points in the direction of greatest increase of \( f \), and it has a magnitude equal to the maximum rate of increase at the point—just like a standard derivative. In particular, if a hill is defined as a height function over a plane \( h(x,y) \), the 2d projection of the gradient at a given location will be a vector in the xy-plane (sort of like an arrow on a map) pointing along the steepest direction. The magnitude of the gradient is the value of this steepest slope.

In particular, this notation is powerful because the gradient product rule looks very similar to the 1d-derivative case:

\[
\nabla (fg) = f\nabla g + g\nabla f
\]

However, the rules for dot products do not turn out to be simple, as illustrated by:

\[
\nabla (\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u} + \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u})
\]

[] Divergence

The divergence of a vector field \( \vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \) is a scalar function that can be represented as:

\[
\text{div } \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot \vec{v}
\]

The divergence is roughly a measure of a vector field's increase in the direction it points; but more accurately a measure of that field's tendency to converge on or repel from a point.

The power of the del notation is shown by the following product rule:

\[
\nabla \cdot (f \vec{v}) = f \nabla \cdot \vec{v} + \vec{v} \cdot \nabla f
\]

The formula for the vector product is slightly less intuitive, because this product is not commutative:
\[ \nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v} \]

[] Curl

The curl of a vector field \( \vec{v}(x, y, z) = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \) is a vector function that can be represented as:

\[
\text{curl } \vec{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z} = \nabla \times \vec{v}
\]

The curl at a point is proportional to the on-axis torque to which a tiny pinwheel would be subjected if it were centered at that point.

The vector product operation can be visualised as a pseudo-determinant:

\[
\nabla \times \vec{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
v_x & v_y & v_z
\end{vmatrix}
\]

Again the power of the notation is shown by the product rule:

\[
\nabla \times (f \vec{v}) = (\nabla f) \times \vec{v} + f \nabla \times \vec{v}
\]

Unfortunately the rule for the vector product does not turn out to be simple:

\[
\nabla \times (\vec{u} \times \vec{v}) = \vec{u} \nabla \cdot \vec{v} - \vec{v} \nabla \cdot \vec{u} + (\vec{v} \cdot \nabla) \vec{u} - (\vec{u} \cdot \nabla) \vec{v}
\]

[] Directional derivative

The directional derivative of a scalar field \( f(x, y, z) \) in the direction \( \vec{a}(x, y, z) = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \) is defined as:

\[
\vec{a} \cdot \text{grad } f = a_x \frac{\partial f}{\partial x} + a_y \frac{\partial f}{\partial y} + a_z \frac{\partial f}{\partial z} = (\vec{a} \cdot \nabla) f
\]

This gives the change of a field \( f \) in the direction of \( \vec{a} \). In operator notation, the element in parentheses can be considered a single coherent unit; fluid dynamics uses this convention extensively, terming it the convective derivative—the "moving" derivative of the fluid.

[] Laplacian
The Laplace operator is a scalar operator that can be applied to either vector or scalar fields; it is defined as:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2$$

The Laplacian is ubiquitous throughout modern mathematical physics, appearing in Laplace's equation, Poisson's equation, the heat equation, the wave equation, and the Schrödinger equation—to name a few.

[] Tensor derivative

Del can also be applied to a vector field with the result being a tensor. The tensor derivative of a vector field $\vec{v}$ is a 9-term second-rank tensor, but can be denoted simply as $\nabla \otimes \vec{v}$, where $\otimes$ represents the dyadic product. This quantity is equivalent to the Jacobian matrix of the vector field with respect to space.

For a small displacement $\delta \vec{r}$, the change in the vector field is given by:

$$\delta \vec{v} = (\nabla \otimes \vec{v}) \cdot \delta \vec{r}$$

[] Second derivatives

When del operates on a scalar or vector, generally a scalar or vector is returned. Because of the diversity of vector products (scalar, dot, cross) one application of del already gives rise to three major derivatives: the gradient (scalar product), divergence (dot product), and curl (cross product). Applying these three sorts of derivatives again to each other gives five possible second derivatives, for a scalar field $f$ or a vector field $\vec{v}$; the use of the scalar Laplacian and vector Laplacian gives two more:

$$\text{div (grad } f) = \nabla \cdot (\nabla f)$$
$$\text{curl (grad } f) = \nabla \times (\nabla f)$$
$$\Delta f = \nabla^2 f$$
$$\text{grad (div } \vec{v}) = \nabla (\nabla \cdot \vec{v})$$
$$\text{div (curl } \vec{v}) = \nabla \cdot (\nabla \times \vec{v})$$
$$\text{curl (curl } \vec{v}) = \nabla \times (\nabla \times \vec{v})$$
$$\Delta \vec{v} = \nabla^2 \vec{v}$$

These are of interest principally because they are not always unique or independent of each other. As long as the functions are well-behaved, two of them are always zero:

$$\text{curl (grad } f) = \nabla \times (\nabla f) = 0$$
$$\text{div (curl } \vec{v}) = \nabla \cdot \nabla \times \vec{v} = 0$$
Two of them are always equal:

\[ \text{div} \, (\text{grad} \, f) = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f \]

The 3 remaining vector derivatives are related by the equation:

\[ \nabla \times \nabla \times \vec{v} = \nabla (\nabla \cdot \vec{v}) - \nabla^2 \vec{v} \]

And one of them can even be expressed with the tensor product, if the functions are well-behaved:

\[ \nabla (\nabla \cdot \vec{v}) = \nabla \cdot (\nabla \otimes \vec{v}) \]

[] Precautions

Most of the above vector properties (except for those that rely explicitly on del's differential properties—for example, the product rule) rely only on symbol rearrangement, and must necessarily hold if del is replaced by any other vector. This is part of the tremendous value gained in representing this operator as a vector in its own right.

Though you can often replace del with a vector and obtain a vector identity, making those identities intuitive, the reverse is not necessarily reliable, because del does not often commute.

A counterexample that relies on del's failure to commute:

\[ (\mathbf{u} \cdot \mathbf{v}) f = (\mathbf{v} \cdot \mathbf{u}) f \]
\[ (\nabla \cdot \mathbf{v}) f \neq (\mathbf{v} \cdot \nabla) f \]
\[ (\nabla \cdot \mathbf{v}) f = (\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}) f = \]
\[ = \frac{\partial v_x}{\partial x} f + \frac{\partial v_y}{\partial y} f + \frac{\partial v_z}{\partial z} f \]
\[ (\mathbf{v} \cdot \nabla) f = (v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}) f = \]
\[ = v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} \]

A counterexample that relies on del's differential properties:

\[ (\nabla x) \times (\nabla y) = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) \times (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) = \]
\[ = (\hat{x} \cdot 1 + \hat{y} \cdot 0 + \hat{z} \cdot 0) \times (\hat{x} \cdot 0 + \hat{y} \cdot 1 + \hat{z} \cdot 0) = \]
\[ = \hat{x} \times \hat{y} = \hat{z} \]
\[(ux) \times (uy) = xy(u \times u) = xy0 = 0\]

Central to these distinctions is the fact that del is not simply a vector; it is a vector operator. Whereas a vector is an object with both a precise numerical magnitude and direction, del does not have a precise value for either until it is allowed to operate on something.

For that reason, identities involving del must be derived from scratch, not derived from pre-existing vector identities.

**Cross product**

In mathematics, the **cross product** is a binary operation on two vectors in a three-dimensional Euclidean space that results in another vector which is perpendicular to the plane containing the two input vectors. The algebra defined by the cross product is neither commutative nor associative. It contrasts with the dot product which produces a scalar result. In many engineering and physics problems, it is desirable to be able to construct a perpendicular vector from two existing vectors, and the cross product provides a means for doing so. The cross product is also useful as a measure of "perpendicularness"—the magnitude of the cross product of two vectors is equal to the product of their magnitudes if they are perpendicular and scales down to zero when they are parallel. The cross product is also known as the **vector product**, or Gibbs vector product.

The cross product is only defined in three or seven dimensions. Like the dot product, it depends on the metric of Euclidean space. Unlike the dot product, it also depends on the choice of orientation or "handedness". Certain features of the cross product can be generalized to other situations. For arbitrary choices of orientation, the cross product must be regarded not as a vector, but as a pseudovector. For arbitrary choices of metric, and in arbitrary dimensions, the cross product can be generalized by the exterior product of vectors, defining a two-form instead of a vector.
The cross-product in respect to a right-handed coordinate system

[] Definition

The cross product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is denoted by \( \mathbf{a} \times \mathbf{b} \). In physics, sometimes the notation \( \mathbf{a} \wedge \mathbf{b} \) is used,[1] though this is avoided in mathematics to avoid confusion with the exterior product.

In a three-dimensional Euclidean space, with a right-handed coordinate system, \( \mathbf{a} \times \mathbf{b} \) is defined as a vector \( \mathbf{c} \) that is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \), with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

The cross product is defined by the formula[2]
\[ \mathbf{a} \times \mathbf{b} = ab \sin \theta \ \hat{n} \]

where \( \theta \) is the measure of the smaller angle between \( \mathbf{a} \) and \( \mathbf{b} \) (\( 0^\circ \leq \theta \leq 180^\circ \)), \( a \) and \( b \) are the magnitudes of vectors \( \mathbf{a} \) and \( \mathbf{b} \), and \( \hat{n} \) is a unit vector perpendicular to the plane containing \( \mathbf{a} \) and \( \mathbf{b} \) in the direction given by the right-hand rule as illustrated. If the vectors \( \mathbf{a} \) and \( \mathbf{b} \) are collinear (i.e., the angle \( \theta \) between them is either \( 0^\circ \) or \( 180^\circ \)), by the above formula, the cross product of \( \mathbf{a} \) and \( \mathbf{b} \) is the zero vector \( \mathbf{0} \).

The direction of the vector \( \hat{n} \) is given by the right-hand rule, where one simply points the forefinger of the right hand in the direction of \( \mathbf{a} \) and the middle finger in the direction of \( \mathbf{b} \). Then, the vector \( \hat{n} \) is coming out of the thumb (see the picture on the right). Using this rule implies that the cross-product is anti-commutative, i.e., \( \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \). By pointing the forefinger toward \( \mathbf{b} \) first, and then pointing the middle finger toward \( \mathbf{a} \), the thumb will be forced in the opposite direction, reversing the sign of the product vector.

Using the cross product requires the handedness of the coordinate system to be taken into account (as explicit in the definition above). If a left-handed coordinate system is used, the direction of the vector \( \hat{n} \) is given by the left-hand rule and points in the opposite direction.

This, however, creates a problem because transforming from one arbitrary reference system to another (e.g., a mirror image transformation from a right-handed to a left-handed coordinate system), should not change the direction of \( \hat{n} \). The problem is clarified by realizing that the cross-product of two vectors is not a (true) vector, but rather a \textit{pseudovector}. See cross product and handedness for more detail.

[] Computing the cross product

[] Coordinate notation

The unit vectors \( \mathbf{i} \), \( \mathbf{j} \), and \( \mathbf{k} \) from the given orthogonal coordinate system satisfy the following equalities:

\[
\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}
\]

Together with the skew-symmetry and bilinearity of the cross product, these three identities are sufficient to determine the cross product of any two vectors. In particular, the following identities are also seen to hold

\[
\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}
\]

\[
\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.
\]

With these rules, the coordinates of the cross product of two vectors can be computed easily, without the need to determine any angles: Let

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} = (a_1, a_2, a_3)
\]
and

\[ \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} = (b_1, b_2, b_3). \]

The cross product can be calculated by distributive cross-multiplication:

\[
\mathbf{a} \times \mathbf{b} = (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k})
\]

\[
\mathbf{a} \times \mathbf{b} = (a_1 \mathbf{i} \times b_1 \mathbf{i}) + (a_1 \mathbf{i} \times b_2 \mathbf{j}) + (a_1 \mathbf{i} \times b_3 \mathbf{k})
\]

\[
+ (a_2 \mathbf{j} \times b_1 \mathbf{i}) + (a_2 \mathbf{j} \times b_2 \mathbf{j}) + (a_2 \mathbf{j} \times b_3 \mathbf{k})
\]

\[
+ (a_3 \mathbf{k} \times b_1 \mathbf{i}) + (a_3 \mathbf{k} \times b_2 \mathbf{j}) + (a_3 \mathbf{k} \times b_3 \mathbf{k}).
\]

Since scalar multiplication is commutative with cross multiplication, the right hand side can be regrouped as

\[
\mathbf{a} \times \mathbf{b} = a_1 b_1 (\mathbf{i} \times \mathbf{i}) + a_1 b_2 (\mathbf{i} \times \mathbf{j}) + a_1 b_3 (\mathbf{i} \times \mathbf{k})
\]

\[
+ a_2 b_1 (\mathbf{j} \times \mathbf{i}) + a_2 b_2 (\mathbf{j} \times \mathbf{j}) + a_2 b_3 (\mathbf{j} \times \mathbf{k})
\]

\[
+ a_3 b_1 (\mathbf{k} \times \mathbf{i}) + a_3 b_2 (\mathbf{k} \times \mathbf{j}) + a_3 b_3 (\mathbf{k} \times \mathbf{k}).
\]

This equation is the sum of nine simple cross products. After all the multiplication is carried out using the basic cross product relationships between \(\mathbf{i}, \mathbf{j}\), and \(\mathbf{k}\) defined above,

\[
\mathbf{a} \times \mathbf{b} = a_1 b_1 (0) + a_1 b_2 (\mathbf{k}) + a_1 b_3 (\mathbf{j})
\]

\[
+ a_2 b_1 (\mathbf{k}) + a_2 b_2 (0) + a_2 b_3 (\mathbf{i})
\]

\[
+ a_3 b_1 (\mathbf{j}) + a_3 b_2 (\mathbf{i}) + a_3 b_3 (0).
\]

This equation can be factored to form

\[
\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).
\]

[] Matrix notation

The definition of the cross product can also be represented by the determinant of a matrix:

\[
\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.
\]

This determinant can be computed using Sarrus' rule. Consider the table

\[
\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{array}
\]

From the first three elements on the first row draw three diagonals sloping downward to the right (for example, the first diagonal would contain \(\mathbf{i}, a_2, \) and \(b_3\)), and from the last three elements on the first row draw three diagonals sloping downward to the left (for example, the first diagonal
would contain \(i, a_3, \) and \(b_2\). Then multiply the elements on each of these six diagonals, and negate the last three products. The cross product would be defined by the sum of these products:

\[
\mathbf{i}a_2 b_3 + \mathbf{j}a_3 b_1 + \mathbf{k}a_1 b_2 - \mathbf{i}a_3 b_2 - \mathbf{j}a_1 b_3 - \mathbf{k}a_2 b_1.
\]

### Properties

#### Geometric meaning

See also: Triple product

The magnitude of the cross product can be interpreted as the positive area of the parallelogram having \(\mathbf{a}\) and \(\mathbf{b}\) as sides (see Figure 1):

\[
A = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta.
\]

Indeed, one can also compute the volume \(V\) of a parallelepiped having \(\mathbf{a}, \mathbf{b}\) and \(\mathbf{c}\) as sides by using a combination of a cross product and a dot product, called scalar triple product (see Figure 2):

\[
V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.
\]
Figure 2 demonstrates that this volume can be found in two ways, showing geometrically that the identity holds that a "dot" and a "cross" can be interchanged without changing the result. That is:

\[ V = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} . \]

Because the magnitude of the cross product goes by the sine of the angle between its arguments, the cross product can be thought of as a measure of "perpendicularness" in the same way that the dot product is a measure of "parallelness". Given two unit vectors, their cross product has a magnitude of 1 if the two are perpendicular and a magnitude of zero if the two are parallel.

[] Algebraic properties

The cross product is anticommutative,

\[ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \]

distributive over addition,

\[ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}), \]

and compatible with scalar multiplication so that

\[ (r \mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (r \mathbf{b}) = r (\mathbf{a} \times \mathbf{b}). \]

It is not associative, but satisfies the Jacobi identity:

\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0. \]

It does not obey the cancellation law:

If \( \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \) and \( \mathbf{a} \neq \mathbf{0} \) then:

\( (\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{c}) = \mathbf{0} \) and, by the distributive law above:

\( \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0} \)

Now, if \( \mathbf{a} \) is parallel to \( \mathbf{b} - \mathbf{c} \), then even if \( \mathbf{a} \neq \mathbf{0} \) it is possible that \( \mathbf{b} - \mathbf{c} \neq \mathbf{0} \) and therefore that \( \mathbf{b} \neq \mathbf{c} \).

However, if both \( \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \) and \( \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \), then it can be concluded that \( \mathbf{b} = \mathbf{c} \). Indeed,

\[ \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0, \]  
\[ \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0} \]

so that \( \mathbf{b} - \mathbf{c} \) is both parallel and perpendicular to the non-zero vector \( \mathbf{a} \). This is only possible if \( \mathbf{b} - \mathbf{c} = \mathbf{0} \).
The distributivity, linearity and Jacobi identity show that $\mathbb{R}^3$ together with vector addition and cross product forms a Lie algebra. In fact, the Lie algebra is that of the real orthogonal group in 3 dimensions, $\text{SO}(3)$.

Further, two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

It follows from the geometrical definition above that the cross product is invariant under rotations about the axis defined by $\mathbf{a} \times \mathbf{b}$.

There is also this property relating cross products and the triple product:

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) \mathbf{a}.$$ 

The cross product obeys this identity under matrix transformations:

$$(M \mathbf{a}) \times (M \mathbf{b}) = (\det M) M^{-T} (\mathbf{a} \times \mathbf{b})$$

where $M$ is a 3 by 3 matrix and $M^{-T}$ is the transpose of the inverse.

The cross product of two vectors in 3-D always lies in the null space of the matrix with the vectors as rows. In other words

$$\mathbf{a} \times \mathbf{b} \in \mathbb{N} \mathbf{S} \left( \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \right)$$

[] Triple product expansion

Main article: Triple product

The triple product expansion, also known as Lagrange's formula, is a formula relating the cross product of three vectors (called the vector triple product) with the dot product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

The mnemonic "BAC minus CAB" is used to remember the order of the vectors in the right hand member. This formula is used in physics to simplify vector calculations. A special case, regarding gradients and useful in vector calculus, is given below.

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - (\nabla \cdot \nabla)\mathbf{f} = \text{grad} (\text{div} \mathbf{f}) - \nabla^2 \mathbf{f}.$$

This is a special case of the more general Laplace-de Rham operator $\Delta = d\delta + \delta d$.

The following identity also relates the cross product and the dot product:
\[ | \mathbf{a} \times \mathbf{b} |^2 + | \mathbf{a} \cdot \mathbf{b} |^2 = | \mathbf{a} |^2 | \mathbf{b} |^2. \]

This is a special case of the multiplicativity \( |\mathbf{v}\mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \) of the norm in the quaternion algebra, and a restriction to \( \mathbb{R}^3 \) of Lagrange's identity.

[] Alternative ways to compute the cross product

[] Quaternions

Further information: quaternions and spatial rotation

The cross product can also be described in terms of quaternions, and this is why the letters \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are a convention for the standard basis on \( \mathbb{R}^3 \): it is thought of as the imaginary quaternions.

For instance, the above given cross product relations among \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) agree with the multiplicative relations among the quaternions \( i, j, \) and \( k \). In general, if a vector \( [a_1, a_2, a_3] \) is represented as the quaternion \( a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \), the cross product of two vectors can be obtained by taking their product as quaternions and deleting the real part of the result. The real part will be the negative of the dot product of the two vectors.

Alternatively and more straightforwardly, using the above identification of the 'purely imaginary' quaternions with \( \mathbb{R}^3 \), the cross product may be thought of as half of the commutator of two quaternions.

[] Conversion to matrix multiplication

A cross product between two vectors (which can only be defined in three-dimensional space) can be rewritten in terms of pure matrix multiplication as the product of a skew-symmetric matrix and a vector, as follows:

\[
\mathbf{a} \times \mathbf{b} = [\mathbf{a}] \times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},
\]

\[
\mathbf{a} \times \mathbf{b} = [\mathbf{b}]^T \times \mathbf{a} = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}
\]

where

\[
[\mathbf{a}] \times \overset{\text{def}}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.
\]
Also, if \( \mathbf{a} \) is itself a cross product:

\[
\mathbf{a} = \mathbf{c} \times \mathbf{d}
\]

then

\[
[\mathbf{a}]_\times = (\mathbf{cd}^T)^T - \mathbf{cd}^T.
\]

This can be generalised to higher dimensions using geometric algebra. In particular in any dimension bivectors from geometric algebra can be identified with skew-symmetric matrices, so the product between a skew-symmetric matrix and vector is just the matrix form of a product between a bivector and vector. In three dimensions bivectors are dual to vectors so the product is equivalent to the cross product, with the bivector instead of its vector dual. In higher dimensions the product can still be calculated but bivectors have more degrees of freedom and are not equivalent to vectors.

This notation is also often much easier to work with, for example, in epipolar geometry.

From the general properties of the cross product follows immediately that

\[
[\mathbf{a}]_\times \mathbf{a} = 0 \quad \text{and} \quad \mathbf{a}^T [\mathbf{a}]_\times = 0
\]

and from fact that \([\mathbf{a}]_\times\) is skew-symmetric it follows that

\[
\mathbf{b}^T [\mathbf{a}]_\times \mathbf{b} = 0.
\]

The above-mentioned triple product expansion (bac-cab rule) can be easily proven using this notation.

The above definition of \([\mathbf{a}]_\times\) means that there is a one-to-one mapping between the set of 3×3 skew-symmetric matrices, also known as the Lie algebra of \(\text{SO}(3)\), and the operation of taking the cross product with some vector \(\mathbf{a}\).

### Index notation

The cross product can alternatively be defined in terms of the Levi-Civita symbol, \(\varepsilon_{ijk}\)

\[
\mathbf{a} \times \mathbf{b} = \mathbf{c} \iff c_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k
\]

where the indices \(i,j,k\) correspond, as in the previous section, to orthogonal vector components. This characterization of the cross product is often expressed more compactly using the Einstein summation convention as
\[ \mathbf{a} \times \mathbf{b} = \mathbf{c} \iff c_i = \varepsilon_{ijk} a_j b_k \]

in which repeated indices are summed from 1 to 3. Note that this representation is another form of the skew-symmetric representation of the cross product:

\[ \varepsilon_{ijk} a_j = [\mathbf{a}]_\times. \]

In classical mechanics: representing the cross-product with the Levi-Civita symbol can cause mechanical-symmetries to be obvious when physical-systems are isotropic in space. (Quick example: consider a particle in a Hooke's Law potential in three-space, free to oscillate in three dimensions; none of these dimensions are "special" in any sense, so symmetries lie in the cross-product-represented angular-momentum which are made clear by the abovementioned Levi-Civita representation). \[\text{[Citation needed]}\]

[] Mnemonic

The word xyzzy can be used to remember the definition of the cross product.

If

\[ \mathbf{a} = \mathbf{b} \times \mathbf{c} \]

where:

\[ \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \]

then:

\[ a_x = b_y c_z - b_z c_y \]
\[ a_y = b_z c_x - b_x c_z \]
\[ a_z = b_x c_y - b_y c_x. \]

The second and third equations can be obtained from the first by simply vertically rotating the subscripts, \( x \to y \to z \to x \). The problem, of course, is how to remember the first equation, and two options are available for this purpose: either to remember the relevant two diagonals of Sarrus's scheme (those containing \( i \)), or to remember the xyzzy sequence.

Since the first diagonal in Sarrus's scheme is just the main diagonal of the above-mentioned \( 3 \times 3 \) matrix, the first three letters of the word xyzzy can be very easily remembered.

[] Cross Visualization
Similarly to the mnemonic device above, a "cross" or X can be visualized between the two vectors in the equation. While this method does not have any real mathematical basis, it may help you to remember the correct Cross Product formula.

If

\[ \mathbf{a} = \mathbf{b} \times \mathbf{c} \]

then:

\[ \mathbf{a} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \times \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \]

If we want to obtain the formula for \( a_x \) we simply drop the \( b_x \) and \( c_x \) from the formula, and take the next two components down -

\[ a_x = \begin{bmatrix} b_y \\ b_z \end{bmatrix} \times \begin{bmatrix} c_y \\ c_z \end{bmatrix} \]

It should be noted that when doing this for \( a_y \) the next two elements down should "wrap around" the matrix so that after the \( z \) component comes the \( x \) component. For clarity, when performing this operation for \( a_y \), the next two components should be \( z \) and \( x \) (in that order). While for \( a_z \) the next two components should be taken as \( x \) and \( y \).

\[ a_y = \begin{bmatrix} b_z \\ b_x \end{bmatrix} \times \begin{bmatrix} c_z \\ c_x \end{bmatrix}, \quad a_z = \begin{bmatrix} b_x \\ b_y \end{bmatrix} \times \begin{bmatrix} c_x \\ c_y \end{bmatrix} \]

For \( a_x \) then, if we visualize the cross operator as pointing from an element on the left to an element on the right, we can take the first element on the left and simply multiply by the element that the cross points to in the right hand matrix. We then subtract the next element down on the left, multiplied by the element that the cross points to here as well. This results in our \( a_x \) formula -

\[ a_x = b_y c_z - b_z c_y \]

We can do this in the same way for \( a_y \) and \( a_z \) to construct their associated formulas.

[] Applications

[] Computational geometry
The cross product can be used to calculate the normal for a triangle or polygon, an operation frequently performed in computer graphics.

In computational geometry of the plane, the cross product is used to determine the sign of the acute angle defined by three points \( p_1=(x_1,y_1), \ p_2=(x_2,y_2)\) and \( p_3=(x_3,y_3) \). It corresponds to the direction of the cross product of the two coplanar vectors defined by the pairs of points \( p_1,p_2\) and \( p_1,p_3\), i.e., by the sign of the expression \( P=(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1) \). In the "right-handed" coordinate system, if the result is 0, the points are collinear; if it is positive, the three points constitute a negative angle of rotation around \( p_2\) from \( p_1\) to \( p_3\), otherwise a positive angle. From another point of view, the sign of \( P\) tells whether \( p_3\) lies to the left or to the right of line \( p_1,p_2\).

[] **Mechanics**

Moment of a force \( \mathbf{F}_B \) applied at point B around point A is given as:

\[
\mathbf{M}_A = \mathbf{r}_{AB} \times \mathbf{F}_B
\]

[] **Other**

The cross product occurs in the formula for the vector operator curl. It is also used to describe the Lorentz force experienced by a moving electrical charge in a magnetic field. The definitions of torque and angular momentum also involve the cross product.

The trick of rewriting a cross product in terms of a matrix multiplication appears frequently in epipolar and multi-view geometry, in particular when deriving matching constraints.

[] **Cross product as an exterior product**

![Cross product as an exterior product](image)

The cross product in relation to the exterior product. In red are the unit normal vector, and the "parallel" unit bivector.

The cross product can be viewed in terms of the exterior product. This view allows for a natural geometric interpretation of the cross product. In exterior calculus the exterior product (or wedge product)
product) of two vectors is a bivector. A bivector is an oriented plane element, in much the same way that a vector is an oriented line element. Given two vectors \(a\) and \(b\), one can view the bivector \(a \wedge b\) as the oriented parallelogram spanned by \(a\) and \(b\). The cross product is then obtained by taking the Hodge dual of the bivector \(a \wedge b\), identifying 2-vectors with vectors:

\[ a \times b = \star (a \wedge b) . \]

This can be thought of as the oriented multi-dimensional element "perpendicular" to the bivector. Only in three dimensions is the result an oriented line element – a vector – whereas, for example, in 4 dimensions the Hodge dual of a bivector is two-dimensional – another oriented plane element. So, in three dimensions only is the cross product of \(a\) and \(b\) the vector dual to the bivector \(a \wedge b\): it is perpendicular to the bivector, with orientation dependent on the coordinate system's handedness, and has the same magnitude relative to the unit normal vector as \(a \wedge b\) has relative to the unit bivector; precisely the properties described above.

[] Cross product and handedness

When measurable quantities involve cross products, the handedness of the coordinate systems used cannot be arbitrary. However, when physics laws are written as equations, it should be possible to make an arbitrary choice of the coordinate system (including handedness). To avoid problems, one should be careful to never write down an equation where the two sides do not behave equally under all transformations that need to be considered. For example, if one side of the equation is a cross product of two vectors, one must take into account that when the handedness of the coordinate system is not fixed a priori, the result is not a (true) vector but a pseudovector. Therefore, for consistency, the other side must also be a pseudovector. [citation needed]

More generally, the result of a cross product may be either a vector or a pseudovector, depending on the type of its operands (vectors or pseudovectors). Namely, vectors and pseudovectors are interrelated in the following ways under application of the cross product:

- vector \( \times \) vector = pseudovector
- vector \( \times \) pseudovector = vector
- pseudovector \( \times \) pseudovector = pseudovector

Because the cross product may also be a (true) vector, it may not change direction with a mirror image transformation. This happens, according to the above relationships, if one of the operands is a (true) vector and the other one is a pseudovector (e.g., the cross product of two vectors). For instance, a vector triple product involving three (true) vectors is a (true) vector.

A handedness-free approach is possible using exterior algebra.

[] Generalizations

There are several ways to generalize the cross product to the higher dimensions.
Lie algebra

Main article: Lie algebra

The cross product can be seen as one of the simplest Lie products, and is thus generalized by Lie algebras, which are axiomatized as binary products satisfying the axioms of multilinearity, skew-symmetry, and the Jacobi identity. Many Lie algebras exist, and their study is a major field of mathematics, called Lie theory.

For example, the Heisenberg algebra gives another Lie algebra structure on $\mathbb{R}^3$. In the basis $\{x,y,z\}$ the product is $[x,y]=z,[x,z]=[y,z]=0$.

Using octonions

Main article: Seven-dimensional cross product

A cross product for 7-dimensional vectors can be obtained in the same way by using the octonions instead of the quaternions. The nonexistence of such cross products of two vectors in other dimensions is related to the result that the only normed division algebras are the ones with dimension 1, 2, 4, and 8.

Wedge product

Main article: Exterior algebra

In general dimension, there is no direct analogue of the binary cross product. There is however the wedge product, which has similar properties, except that the wedge product of two vectors is now a 2-vector instead of an ordinary vector. As mentioned above, the cross product can be interpreted as the wedge product in three dimensions after using Hodge duality to identify 2-vectors with vectors.

The wedge product and dot product can be combined to form the Clifford product.

Multilinear algebra

In the context of multilinear algebra, the cross product can be seen as the (1,2)-tensor (a mixed tensor) obtained from the 3-dimensional volume form, [note 1] a (0,3)-tensor, by raising an index.

In detail, the 3-dimensional volume form defines a product $V \times V \times V \rightarrow \mathbb{R}$ by taking the determinant of the matrix given by these 3 vectors. By duality, this is equivalent to a function $V \times V \rightarrow V^*$ (fixing any two inputs gives a function $V \rightarrow \mathbb{R}$ by evaluating on the third input) and in the presence of an inner product (such as the dot product; more generally, a non-degenerate bilinear form), we have an isomorphism $V \rightarrow V^*$ and thus this yields a map $V \times V \rightarrow V$ which is the cross product: a (0,3)-tensor (3 vector inputs, scalar output) has been transformed into a (1,2)-tensor (2 vector inputs, 1 vector output) by "raising an index".
Translating the above algebra into geometry, the function "volume of the parallelepiped defined by \((a,b,\ldots)\)" (where the first two vectors are fixed and the last is an input), which defines a function \(V \to \mathbf{R}\), can be represented uniquely as the dot product with a vector: this vector is the cross product \(a \times b\). From this perspective, the cross product is defined by the scalar triple product, \(\text{Vol}(a,b,c) = (a \times b) \cdot c\).

In the same way, in higher dimensions one may define generalized cross products by raising indices of the \(n\)-dimensional volume form, which is a \((0,n)\)-tensor. The most direct generalizations of the cross product are to define either:

- a \((1,n-1)\)-tensor, which takes as input \(n-1\) vectors, and gives as output 1 vector – an \((n-1)\)-ary vector-valued product, or
- a \((n-2,2)\)-tensor, which takes as input 2 vectors and gives as output skew-symmetric tensor of rank \(n-2\) – a binary product with rank \(n-2\) tensor values. One can also define \((k,n-k)\)-tensors for other \(k\).

These products are all multilinear and skew-symmetric, and can be defined in terms of the determinant and parity.

The \((n-1)\)-ary product can be described as follows: given \(n-1\) vectors \(v_1, \ldots, v_{n-1}\) in \(\mathbf{R}^n\), define their generalized cross product \(v_n = v_1 \times \cdots \times v_{n-1}\) as:

- perpendicular to the hyperplane defined by the \(v_i\),
- magnitude is the volume of the parallelotope defined by the \(v_i\), which can be computed as the Gram determinant of the \(v_i\),
- oriented so that \(v_1, \ldots, v_n\) is positively oriented.

This is the unique multilinear, alternating product which evaluates to \(e_1 \times \cdots \times e_{n-1} = e_n, e_2 \times \cdots \times e_n = e_1\) and so forth for cyclic permutations of indices.

In coordinates, one can give a formula for this \(n\)-ary analogue of the cross product in \(\mathbf{R}^{n+1}\) by:

\[
\bigwedge(v_1, \ldots, v_n) = \begin{vmatrix}
v_1^1 & \cdots & v_1^{n+1} \\
\vdots & \ddots & \vdots \\
v_n^1 & \cdots & v_n^{n+1} \\
e_1 & \cdots & e_{n+1}
\end{vmatrix}.
\]

This formula is identical in structure to the determinant formula for the normal cross product in \(\mathbf{R}^3\) except that the row of basis vectors is the last row in the determinant rather than the first. The reason for this is to ensure that the ordered vectors \((v_1, \ldots, v_n, \bigwedge(v_1, \ldots, v_n))\) have a positive orientation with respect to \((e_1, \ldots, e_{n+1})\). If \(n\) is even, this modification leaves the value unchanged, so this convention agrees with the normal definition of the binary product. In the case that \(n\) is odd, however, the distinction must be kept. This \(n\)-ary form enjoys many of the same properties as the vector cross product: it is alternating and linear in its arguments, it is perpendicular to each
argument, and its magnitude gives the hypervolume of the region bounded by the arguments. And just like the vector cross product, it can be defined in a coordinate independent way as the Hodge dual of the wedge product of the arguments.

[] History

In 1773, Joseph Louis Lagrange introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir William Rowan Hamilton introduced the quaternion product, and with it the terms "vector" and "scalar". Given two quaternions \([0, \mathbf{u}]\) and \([0, \mathbf{v}]\), where \(\mathbf{u}\) and \(\mathbf{v}\) are vectors in \(\mathbb{R}^3\), their quaternion product can be summarized as \([-\mathbf{u} \cdot \mathbf{v}, \mathbf{u} \times \mathbf{v}]\). James Clerk Maxwell used Hamilton's quaternion tools to develop his famous electromagnetism equations, and for this and other reasons quaternions for a time were an essential part of physics education.

However, Oliver Heaviside in England and Josiah Willard Gibbs in Connecticut felt that quaternion methods were too cumbersome, often requiring the scalar or vector part of a result to be extracted. Thus, about forty years after the quaternion product, the dot product and cross product were introduced—to heated opposition. Pivotal to (eventual) acceptance was the efficiency of the new approach, allowing Heaviside to reduce the equations of electromagnetism from Maxwell's original 20 to the four commonly seen today.

Largely independent of this development, and largely unappreciated at the time, Hermann Grassmann created a geometric algebra not tied to dimension two or three, with the exterior product playing a central role. William Kingdon Clifford combined the algebras of Hamilton and Grassmann to produce Clifford algebra, where in the case of three-dimensional vectors the bivector produced from two vectors dualizes to a vector, thus reproducing the cross product.

The cross notation, which began with Gibbs, inspired the name "cross product". Originally appearing in privately published notes for his students in 1881 as *Elements of Vector Analysis*, Gibbs's notation—and the name—later reached a wider audience through *Vector Analysis* (Gibbs/Wilson), a textbook by a former student. Edwin Bidwell Wilson rearranged material from Gibbs's lectures, together with material from publications by Heaviside, Föppps, and Hamilton. He divided vector analysis into three parts:

First, that which concerns addition and the scalar and vector products of vectors. Second, that which concerns the differential and integral calculus in its relations to scalar and vector functions. Third, that which contains the theory of the linear vector function.

Two main kinds of vector multiplications were defined, and they were called as follows:

- The **direct, scalar**, or **dot** product of two vectors
- The **skew, vector**, or **cross** product of two vectors

Several kinds of triple products and products of more than three vectors were also examined. The above mentioned triple product expansion was also included.
Dot product

In mathematics, the dot product is an algebraic operation that takes two equal-length sequences of numbers (usually coordinate vectors) and returns a single number obtained by multiplying corresponding entries and adding up those products. The name is derived from the dot that is often used to designate this operation; the alternative name scalar product emphasizes the scalar (rather than vector) nature of the result.

The principal use of this product is the inner product in a Euclidean vector space: when two vectors are expressed on an orthonormal basis, the dot product of their coordinate vectors gives their inner product. For this geometric interpretation, scalars must be taken to be real numbers; while the dot product can be defined in a more general setting (for instance with complex numbers as scalars) many properties would be different. The dot product contrasts (in three dimensional space) with the cross product, which produces a vector as result.

[] Definition

The dot product of two vectors \( \mathbf{a} = [a_1, a_2, \ldots, a_n] \) and \( \mathbf{b} = [b_1, b_2, \ldots, b_n] \) is defined as:

\[
\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n
\]

where \( \Sigma \) denotes summation notation and \( n \) is the dimension of the vectors.

In dimension 2, the dot product of vectors \( [a,b] \) and \( [c,d] \) is \( ac + bd \). Similarly, in a dimension 3, the dot product of vectors \( [a,b,c] \) and \( [d,e,f] \) is \( ad + be + cf \). For example, the dot product of two three-dimensional vectors \( [1, 3, -5] \) and \( [4, -2, -1] \) is

\[
[1, 3, -5] \cdot [4, -2, -1] = 1 \times 4 + 3 \times -2 + -5 \times -1 = 3.
\]

The dot product can also be obtained via transposition and matrix multiplication as follows:

\[
\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b},
\]

where both vectors are interpreted as column vectors, and \( \mathbf{a}^T \) denotes the transpose of \( \mathbf{a} \), in other words the corresponding row vector.

[] Geometric interpretation
\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos(\theta). \]

\(|\mathbf{A}| \cos(\theta)\) is the scalar projection of \(\mathbf{A}\) onto \(\mathbf{B}\).

In Euclidean geometry, the dot product, length, and angle are related. For a vector \(\mathbf{a}\), the dot product \(\mathbf{a} \cdot \mathbf{a}\) is the square of the length of \(\mathbf{a}\), or

\[ |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} \]

where \(|\mathbf{a}|\) denotes the length (magnitude) of \(\mathbf{a}\). More generally, if \(\mathbf{b}\) is another vector

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \]

where \(|\mathbf{a}|\) and \(|\mathbf{b}|\) denote the length of \(\mathbf{a}\) and \(\mathbf{b}\) and \(\theta\) is the angle between them.

Thus, given two vectors, the angle between them can be found by rearranging the above formula:

\[ \theta = \arccos \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \right). \]

The cosine of the angle is returned because each vector \(\mathbf{a}\) and \(\mathbf{b}\) become unit vectors when we divide by the length.

\[ \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \] is the unit vector.

The terminal points of both unit vectors lie on the unit circle. The unit circle is where the trigonometric values for the six trig functions are found. After substitution, the first vector component is cosine and the second vector component is sine, i.e. \((\cos x, \sin x)\) for some angle \(x\). The dot product of the two unit vectors then takes \(<\cos x, \sin x><\cos y, \sin y>\) for angles \(x, y\) and returns \((\cos x)(\cos y) + (\sin x)(\sin y) = \cos(x - y)\) where \(x - y = \theta\).
As the cosine of 90° is zero, the dot product of two orthogonal vectors is always zero. Moreover, two vectors can be considered orthogonal if and only if their dot product is zero, and they have non-null length. This property provides a simple method to test the condition of orthogonality.

Sometimes these properties are also used for defining the dot product, especially in 2 and 3 dimensions; this definition is equivalent to the above one. For higher dimensions the formula can be used to define the concept of angle.

The geometric properties rely on the basis being orthonormal, i.e. composed of pairwise perpendicular vectors with unit length.

[] Scalar projection

If both \( \mathbf{a} \) and \( \mathbf{b} \) have length one (i.e. they are unit vectors), their dot product simply gives the cosine of the angle between them.

If only \( \mathbf{b} \) is a unit vector, then the dot product \( \mathbf{a} \cdot \mathbf{b} \) gives \( |\mathbf{a}| \cos(\theta) \), i.e. the magnitude of the projection of \( \mathbf{a} \) in the direction of \( \mathbf{b} \), with a minus sign if the direction is opposite. This is called the scalar projection of \( \mathbf{a} \) onto \( \mathbf{b} \), or scalar component of \( \mathbf{a} \) in the direction of \( \mathbf{b} \) (see figure). This property of the dot product has several useful applications (for instance, see next section).

If neither \( \mathbf{a} \) nor \( \mathbf{b} \) is a unit vector, then the magnitude of the projection of \( \mathbf{a} \) in the direction of \( \mathbf{b} \), for example, would be \( \mathbf{a} \cdot (\mathbf{b} / |\mathbf{b}|) \) as the unit vector in the direction of \( \mathbf{b} \) is \( \mathbf{b} / |\mathbf{b}| \).

[] Rotation

A rotation of the orthonormal basis in terms of which vector \( \mathbf{a} \) is represented is obtained with a multiplication of \( \mathbf{a} \) by a rotation matrix \( \mathbf{R} \). This matrix multiplication is just a compact representation of a sequence of dot products.

For instance, let

- \( B_1 = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \) and \( B_2 = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \) be two different orthonormal bases of the same space \( \mathbb{R}^3 \), with \( B_2 \) obtained by just rotating \( B_1 \),
- \( \mathbf{a}_1 = (a_x, a_y, a_z) \) represent vector \( \mathbf{a} \) in terms of \( B_1 \),
- \( \mathbf{a}_2 = (a_u, a_v, a_w) \) represent the same vector in terms of the rotated basis \( B_2 \),
- \( \mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1 \) be the rotated basis vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) represented in terms of \( B_1 \).

Then the rotation from \( B_1 \) to \( B_2 \) is performed as follows:

\[
\mathbf{a}_2 = \mathbf{R} \mathbf{a}_1 = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{a}_1 \\ \mathbf{v}_1 \cdot \mathbf{a}_1 \\ \mathbf{w}_1 \cdot \mathbf{a}_1 \end{bmatrix} = \begin{bmatrix} a_u \\ a_v \\ a_w \end{bmatrix}.
\]
Notice that the rotation matrix $\mathbf{R}$ is assembled by using the rotated basis vectors $\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1$ as its rows, and these vectors are unit vectors. By definition, $\mathbf{R}\mathbf{a}_1$ consists of a sequence of dot products between each of the three rows of $\mathbf{R}$ and vector $\mathbf{a}_1$. Each of these dot products determines a scalar component of $\mathbf{a}$ in the direction of a rotated basis vector (see previous section).

If $\mathbf{a}_1$ is a row vector, rather than a column vector, then $\mathbf{R}$ must contain the rotated basis vectors in its columns, and must post-multiply $\mathbf{a}_1$:

$$\mathbf{a}_2 = \mathbf{a}_1 \mathbf{R} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \begin{bmatrix} u_1 \cdot \mathbf{a}_1 & v_1 \cdot \mathbf{a}_1 & w_1 \cdot \mathbf{a}_1 \end{bmatrix} = \begin{bmatrix} a_u & a_v \end{bmatrix}$$

[] Physics

In physics, magnitude is a scalar in the physical sense, i.e. a physical quantity independent of the coordinate system, expressed as the product of a numerical value and a physical unit, not just a number. The dot product is also a scalar in this sense, given by the formula, independent of the coordinate system. The formula in terms of coordinates is evaluated with not just numbers, but numbers times units. Therefore, although it relies on the basis being orthonormal, it does not depend on scaling.

Example:

- Mechanical work is the dot product of force and displacement vectors.

[] Properties

The following properties hold if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are real vectors and $r$ is a scalar.

The dot product is commutative:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$  

The dot product is distributive over vector addition:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$  

The dot product is bilinear:

$$\mathbf{a} \cdot (r \mathbf{b} + \mathbf{c}) = r(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c}).$$

When multiplied by a scalar value, dot product satisfies:
\( (c_1 \mathbf{a}) \cdot (c_2 \mathbf{b}) = (c_1 c_2)(\mathbf{a} \cdot \mathbf{b}) \)

(these last two properties follow from the first two).

Two non-zero vectors \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \).

Unlike multiplication of ordinary numbers, where if \( ab = ac \), then \( b \) always equals \( c \) unless \( a \) is zero, the dot product does not obey the cancellation law:

If \( \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \) and \( \mathbf{a} \neq \mathbf{0} \), then we can write: \( \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0 \) by the distributive law; the result above says this just means that \( \mathbf{a} \) is perpendicular to \( (\mathbf{b} - \mathbf{c}) \), which still allows \( (\mathbf{b} - \mathbf{c}) \neq \mathbf{0} \), and therefore \( \mathbf{b} \neq \mathbf{c} \).

Provided that the basis is orthonormal, the dot product is invariant under isometric changes of the basis: rotations, reflections, and combinations, keeping the origin fixed. The above mentioned geometric interpretation relies on this property. In other words, for an orthonormal space with any number of dimensions, the dot product is invariant under a coordinate transformation based on an orthogonal matrix. This corresponds to the following two conditions:

- The new basis is again orthonormal (i.e., it is orthonormal expressed in the old one).
- The new base vectors have the same length as the old ones (i.e., unit length in terms of the old basis).

If \( \mathbf{a} \) and \( \mathbf{b} \) are functions, then the derivative of \( \mathbf{a} \cdot \mathbf{b} \) is \( \mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}' \)

\[ \text{Triple product expansion} \]

Main article: Triple product

This is a very useful identity (also known as Lagrange's formula) involving the dot- and cross-products. It is written as

\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \]

which is easier to remember as "BAC minus CAB", keeping in mind which vectors are dotted together. This formula is commonly used to simplify vector calculations in physics.

\[ \text{Proof of the geometric interpretation} \]

Note: This proof is shown for 3-dimensional vectors, but is readily extended to \( n \)-dimensional vectors.

Consider a vector

\[ \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \]
Repeated application of the Pythagorean theorem yields for its length \( v \)
\[
v^2 = v_1^2 + v_2^2 + v_3^2.
\]
But this is the same as
\[
\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2.
\]
so we conclude that taking the dot product of a vector \( \mathbf{v} \) with itself yields the squared length of the vector.

**Lemma 1**
\[
\mathbf{v} \cdot \mathbf{v} = v^2.
\]
Now consider two vectors \( \mathbf{a} \) and \( \mathbf{b} \) extending from the origin, separated by an angle \( \theta \). A third vector \( \mathbf{c} \) may be defined as
\[
\mathbf{c} \overset{\text{def}}{=} \mathbf{a} - \mathbf{b}.
\]
creating a triangle with sides \( a, b, \) and \( c \). According to the law of cosines, we have
\[
c^2 = a^2 + b^2 - 2ab \cos \theta.
\]
Substituting dot products for the squared lengths according to Lemma 1, we get
\[
\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2ab \cos \theta. \quad (1)
\]
But as \( \mathbf{c} \equiv \mathbf{a} - \mathbf{b} \), we also have
\[
\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}),
\]
which, according to the distributive law, expands to
\[
\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b}). \quad (2)
\]
Merging the two \( \mathbf{c} \cdot \mathbf{c} \) equations, \( (1) \) and \( (2) \), we obtain
\[
\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2ab \cos \theta.
\]
Subtracting \( \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \) from both sides and dividing by \(-2\) leaves
\[
\mathbf{a} \cdot \mathbf{b} = ab \cos \theta.
\]
Q.E.D.
[] Generalization

The inner product generalizes the dot product to abstract vector spaces and is usually denoted by \( \langle \mathbf{a}, \mathbf{b} \rangle \). Due to the geometric interpretation of the dot product the norm \( \| \mathbf{a} \| \) of a vector \( \mathbf{a} \) in such an inner product space is defined as

\[
\| \mathbf{a} \| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}
\]

such that it generalizes length, and the angle \( \theta \) between two vectors \( \mathbf{a} \) and \( \mathbf{b} \) by

\[
\cos \theta = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\| \mathbf{a} \| \| \mathbf{b} \|}.
\]

In particular, two vectors are considered orthogonal if their inner product is zero

\[
\langle \mathbf{a}, \mathbf{b} \rangle = 0.
\]

For vectors with complex entries, using the given definition of the dot product would lead to quite different geometric properties. For instance the dot product of a vector with itself can be an arbitrary complex number, and can be zero without the vector being the zero vector; this in turn would have severe consequences for notions like length and angle. Many geometric properties can be salvaged, at the cost of giving up the symmetric and bilinear properties of the scalar product, by alternatively defining

\[
\mathbf{a} \cdot \mathbf{b} = \sum a_i \overline{b_i}
\]

where \( b_i \) is the complex conjugate of \( b_i \). Then the scalar product of any vector with itself is a non-negative real number, and it is nonzero except for the zero vector. However this scalar product is not linear in \( \mathbf{b} \) (but rather conjugate linear), and the scalar product is not symmetric either, since

\[
\mathbf{a} \cdot \mathbf{b} = \overline{\mathbf{b} \cdot \mathbf{a}}.
\]

This type of scalar product is nevertheless quite useful, and leads to the notions of Hermitian form and of general inner product spaces.

The Frobenius inner product generalizes the dot product to matrices. It is defined as the sum of the products of the corresponding components of two matrices having the same size.

[] Generalization to tensors

The dot product between a tensor of order \( n \) and a tensor of order \( m \) is a tensor of order \( n+m-2 \). The dot product is calculated by multiplying and summing across a single index in both tensors.
If $\mathbf{A}$ and $\mathbf{B}$ are two tensors with element representation $A_{ij...}^{k\ell...}$ and $B_{mn...}^{p...i}$, the elements of the dot product $\mathbf{A} \cdot \mathbf{B}$ are given by

$$A_{ij...}^{k\ell...} B_{mn...}^{p...i} = \sum_{i=1}^{n} A_{ij...}^{k\ell...} B_{mn...}^{p...i}$$

This definition naturally reduces to the standard vector dot product when applied to vectors, and matrix multiplication when applied to matrices.

Occasionally, a double dot product is used to represent multiplying and summing across two indices. The double dot product between two 2nd order tensors is a scalar.

**Vector-valued function**

![Graph of vector-valued function](image)

A graph of the vector-valued function $\mathbf{r}(t) = \langle 2 \cos t, 4 \sin t, t \rangle$ indicating a range of solutions and the vector when evaluated near $t = 19.5$

A vector-valued function also referred to as a vector function is a mathematical function of one or more variables whose range is a set of multidimensional vectors. Often the input of a vector-valued function is a scalar, but in general the input can be a vector of both complex or real variables.

[] Example

A common example of a vector valued function is one that depends on a single real number parameter $t$, often representing time, producing a vector $\mathbf{v}(t)$ as the result. In terms of the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of Cartesian 3-space, these specific type of vector-valued functions are given by expressions such as
\[ \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \]
\[ \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \]

where \( f(t) \), \( g(t) \) and \( h(t) \) are the **coordinate functions** of the parameter \( t \). The vector \( \mathbf{r}(t) \) has its tail at the origin and its head at the coordinates evaluated by the function.

The vector shown in the graph to the right is the evaluation of the function near \( t = 19.5 \) (between \( 6\pi \) and \( 6.5\pi \); i.e., somewhat more than 3 rotations). The spiral is the path traced by the tip of the vector as \( t \) increases from zero through \( 8\pi \).

Vector functions can also be referred to in a different notation:

\[ \mathbf{r}(t) = \langle f(t), g(t) \rangle \]
\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \]

### [] Properties

The domain of a vector-valued function is the intersection of the domain of the functions \( f \), \( g \), and \( h \).

### [] Derivative of a vector function

Many vector-valued functions, like scalar-valued functions, can be differentiated by simply differentiating the components in the Cartesian coordinate system. Thus, if

\[ \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \]

is a vector-valued function, then

\[ \frac{d\mathbf{r}(t)}{dt} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \]

The vector derivative admits the following physical interpretation: if \( \mathbf{r}(t) \) represents the position of a particle, then the derivative is the velocity of the particle

\[ \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}. \]

Likewise, the derivative of the velocity is the acceleration

\[ \frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t). \]
Partial derivative

The partial derivative of a vector function \( \mathbf{a} \) with respect to a scalar variable \( q \) is defined as\(^{[1]}\)

\[
\frac{\partial \mathbf{a}}{\partial q} = \sum_{i=1}^{3} \frac{\partial a_i}{\partial q} \mathbf{e}_i
\]

where \( a_i \) is the scalar component of \( \mathbf{a} \) in the direction of \( \mathbf{e}_i \). It is also called the direction cosine of \( \mathbf{a} \) and \( \mathbf{e}_i \) or their dot product. The vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) form an orthonormal basis fixed in the reference frame in which the derivative is being taken.

Ordinary derivative

If \( \mathbf{a} \) is regarded as a vector function of a single scalar variable, such as time \( t \), then the equation above reduces to the first ordinary time derivative of \( \mathbf{a} \) with respect to \( t \),\(^{[1]}\)

\[
\frac{d\mathbf{a}}{dt} = \sum_{i=1}^{3} \frac{da_i}{dt} \mathbf{e}_i.
\]

Total derivative

If the vector \( \mathbf{a} \) is a function of a number \( n \) of scalar variables \( q_r \) (\( r = 1, \ldots, n \)), and each \( q_r \) is only a function of time \( t \), then the ordinary derivative of \( \mathbf{a} \) with respect to \( t \) can be expressed, in a form known as the total derivative, as\(^{[1]}\)

\[
\frac{d\mathbf{a}}{dt} = \sum_{r=1}^{n} \frac{\partial \mathbf{a}}{\partial q_r} \frac{dq_r}{dt} + \frac{\partial \mathbf{a}}{\partial t}.
\]

Some authors prefer to use capital \( D \) to indicate the total derivative operator, as in \( D/dt \). The total derivative differs from the partial time derivative in that the total derivative accounts for changes in \( \mathbf{a} \) due to the time variance of the variables \( q_r \).

Reference frames

Whereas for scalar-valued functions there is only a single possible reference frame, to take the derivative of a vector-valued function requires the choice of a reference frame (at least when a fixed Cartesian coordinate system is not implied as such). Once a reference frame has been chosen, the derivative of a vector-valued function can be computed using techniques similar to those for computing derivatives of scalar-valued functions. A different choice of reference frame will, in general, produce a different derivative function. The derivative functions in different reference frames have a specific kinematical relationship.
[] Derivative of a vector function with nonfixed bases

The above formulas for the derivative of a vector function rely on the assumption that the basis vectors $e_1, e_2, e_3$ are constant, that is, fixed in the reference frame in which the derivative of $a$ is being taken, and therefore the $e_1, e_2, e_3$ each has a derivative of identically zero. This often holds true for problems dealing with vector fields in a fixed coordinate system, or for simple problems in physics. However, many complex problems involve the derivative of a vector function in multiple moving reference frames, which means that the basis vectors will not necessarily be constant. In such a case where the basis vectors $e_1, e_2, e_3$ are fixed in reference frame $E$, but not in reference frame $N$, the more general formula for the ordinary time derivative of a vector in reference frame $N$ is$^1$

$$\frac{N da}{dt} = \sum_{i=1}^{3} \frac{da_i}{dt} e_i + \sum_{i=1}^{3} a_i \frac{N de_i}{dt}$$

where the superscript $N$ to the left of the derivative operator indicates the reference frame in which the derivative is taken. As shown previously, the first term on the right hand side is equal to the derivative of $a$ in the reference frame where $e_1, e_2, e_3$ are constant, reference frame $E$. It also can be shown that the second term on the right hand side is equal to the relative angular velocity of the two reference frames cross multiplied with the vector $a$ itself.$^1$ Thus, after substitution, the formula relating the derivative of a vector function in two reference frames is$^1$

$$\frac{N da}{dt} = \frac{E da}{dt} + N \omega^E \times a$$

where $N \omega^E$ is the angular velocity of the reference frame $E$ relative to the reference frame $N$.

One common example where this formula is used is to find the velocity of a space-borne object, such as a rocket, in the inertial reference frame using measurements of the rocket's velocity relative to the ground. The velocity $N v^R$ in inertial reference frame $N$ of a rocket $R$ located at position $r^R$ can be found using the formula

$$\frac{N d}{dt} (r^R) = \frac{E d}{dt} (r^R) + N \omega^E \times r^R.$$

where $N \omega^E$ is the angular velocity of the Earth relative to the inertial frame $N$. Since velocity is the derivative of position, $N v^R$ and $E v^R$ are the derivatives of $r^R$ in reference frames $N$ and $E$, respectively. By substitution,

$$N v^R = E v^R + N \omega^E \times r^R$$

where $E v^R$ is the velocity vector of the rocket as measured from a reference frame $E$ that is fixed to the Earth.
[ ] Derivative and vector multiplication

The derivative of the products of vector functions behaves similarly to the derivative of the products of scalar functions.[2] Specifically, in the case of scalar multiplication of a vector, if \( p \) is a scalar variable function of \( q \),[1]

\[
\frac{\partial}{\partial q}(pa) = \frac{\partial p}{\partial q}a + p\frac{\partial a}{\partial q}.
\]

In the case of dot multiplication, for two vectors \( a \) and \( b \) that are both functions of \( q \),[1]

\[
\frac{\partial}{\partial q}(a \cdot b) = \frac{\partial a}{\partial q} \cdot b + a \cdot \frac{\partial b}{\partial q}.
\]

Similarly, the derivative of the cross product of two vector functions is[1]

\[
\frac{\partial}{\partial q}(a \times b) = \frac{\partial a}{\partial q} \times b + a \times \frac{\partial b}{\partial q}.
\]

Vector field

A portion of the vector field \((\sin y, \sin x)\)

In mathematics a vector field is a construction in vector calculus which associates a vector to every point in a subset of Euclidean space.
Vector fields are often used in physics to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.

In the rigorous mathematical treatment, (tangent) vector fields are defined on manifolds as sections of a manifold's tangent bundle. They are one kind of tensor field on the manifold.

**[] Definition**

**[] Vector fields on subsets of Euclidean space**

Given a subset $S$ in $\mathbb{R}^n$, a vector field is represented by a vector-valued function $V : S \to \mathbb{R}^n$ in standard Cartesian coordinates $(x_1, ..., x_n)$. If $S$ is an open set, then $V$ is a continuous function provided that each component of $V$ is continuous, and more generally $V$ is $C^k$ vector field if each component $V$ is $k$ times continuously differentiable.

A vector field can be visualized as an $n$-dimensional space with an $n$-dimensional vector attached to each point. Given two $C^k$-vector fields $V, W$ defined on $S$ and a real valued $C^k$-function $f$ defined on $S$, the two operations scalar multiplication and vector addition

\[
(fV)(p) := f(p)V(p)
\]

\[
(V + W)(p) := V(p) + W(p)
\]

define the module of $C^k$-vector fields over the ring of $C^k$-functions.

**[] Coordinate transformation law**

In physics, a vector is additionally distinguished by how its coordinates change when one measures the same vector with respect to a different background coordinate system. The transformation properties of vectors distinguish a vector as a geometrically distinct entity from a simple list of scalars, or from a covector.

Thus, suppose that $(x_1, ..., x_n)$ is a choice of Cartesian coordinates, in terms of which the coordinates of the vector $V$ are

\[
V_x = (V_{1,x}, \ldots, V_{n,x})
\]

and suppose that $(y_1, ..., y_n)$ are $n$ functions of the $x_i$ defining a different coordinate system. Then the coordinates of the vector $V$ in the new coordinates are required to satisfy the transformation law

\[
V_{i,y} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} V_{j,x} \tag{I}
\]
Such a transformation law is called contravariant. A similar transformation law characterizes vector fields in physics: specifically, a vector field is a specification of $n$ functions in each coordinate system subject to the transformation law (1) relating the different coordinate systems.

Vector fields are thus contrasted with scalar fields, which associate a number or scalar to every point in space, and are also contrasted with simple lists of scalar fields, which do not transform under coordinate changes.

[] Vector fields on manifolds

![A vector field on a sphere](image)

Given a manifold $M$, a **vector field** on $M$ is a *continuous* assignment to every point of $M$ a tangent vector to $M$ at that point. That is, for each $x$ in $M$, we have a tangent vector $v(x)$ in $T_xM$ such that the map sending a point to the appropriate tangent vector is a continuous function from the manifold to the total space of its tangent bundle. More precisely, a vector field is a section of the tangent bundle $TM$. If this section is continuous/differentiable/smooth/analytic, then we call the vector field continuous/differentiable/smooth/analytic. It is important to note that these properties are invariant under the change of coordinates formula, and thus can be detected by computing the local representation in any continuous/differentiable/smooth/analytic chart.

The collection of all vector fields on $M$ is often denoted by $\Gamma(TM)$ or $C^\infty(M,TM)$ (especially when thinking of vector fields as sections); the collection of all smooth vector fields is sometimes also denoted by $\mathfrak{X}(M)$ (a fraktur "$X$").
The flow field around an airplane is a vector field in $\mathbb{R}^3$, here visualized by bubbles that follow the streamlines showing a wingtip vortex.

- A vector field for the movement of air on Earth will associate for every point on the surface of the Earth a vector with the wind speed and direction for that point. This can be drawn using arrows to represent the wind; the length (magnitude) of the arrow will be an indication of the wind speed. A "high" on the usual barometric pressure map would then act as a source (arrows pointing away), and a "low" would be a sink (arrows pointing towards), since air tends to move from high pressure areas to low pressure areas.
- Velocity field of a moving fluid. In this case, a velocity vector is associated to each point in the fluid.
- Streamlines, Streaklines and Pathlines are 3 types of lines that can be made from vector fields. They are:

  streaklines — as revealed in wind tunnels using smoke.
  streamlines (or fieldlines)— as a line depicting the instantaneous field at a given time.
  pathlines — showing the path that a given particle (of zero mass) would follow.

- Magnetic fields. The fieldlines can be revealed using small iron filings.
- Maxwell's equations allow us to use a given set of initial conditions to deduce, for every point in Euclidean space, a magnitude and direction for the force experienced by a charged test particle at that point; the resulting vector field is the electromagnetic field.
- A gravitational field generated by any massive object is also a vector field. For example, the gravitational field vectors for a spherically symmetric body would all point towards the sphere's center with the magnitude of the vectors reducing as radial distance from the body increases.

[] Gradient field

Vector fields can be constructed out of scalar fields using the gradient operator (denoted by the del: $\nabla$) which gives rise to the following definition.

A vector field $V$ defined on a set $S$ is called a gradient field or a conservative field if there exists a real-valued function (a scalar field) $f$ on $S$ such that

$$V = \nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n} \right).$$
The associated flow is called the **gradient flow**, and is used in the method of gradient descent.

The path integral along any closed curve \( \gamma (\gamma(0) = \gamma(1)) \) in a gradient field is zero:

\[
\int_{\gamma} \langle V(x), \, dx \rangle = \int_{\gamma} \langle \nabla f(x), \, dx \rangle = f(\gamma(1)) - f(\gamma(0)).
\]

[[]] **Central field**

A \( C^\infty \)-vector field over \( \mathbb{R}^n \setminus \{0\} \) is called a **central field** if

\[
V(T(p)) = T(V(p)) \quad (T \in O(n, \mathbb{R}))
\]

where \( O(n, \mathbb{R}) \) is the orthogonal group. We say central fields are invariant under orthogonal transformations around 0.

The point 0 is called the **center** of the field.

Since orthogonal transformations are actually rotations and reflections, the invariance conditions mean that vectors of a central field are always directed towards, or away from, 0; this is an alternate (and simpler) definition. A central field is always a gradient field, since defining it on one semiaxis and integrating gives an antigradient.

[[]] **Operations on vector fields**

[[]] **Line integral**

Main article: Line integral

A common technique in physics is to integrate a vector field along a curve: to determine a line integral. Given a particle in a gravitational vector field, where each vector represents the force acting on the particle at a given point in space, the line integral is the work done on the particle when it travels along a certain path.

The line integral is constructed analogously to the Riemann integral and it exists if the curve is rectifiable (has finite length) and the vector field is continuous.

Given a vector field \( V \) and a curve \( \gamma \) parametrized by \([0, 1]\) the line integral is defined as

\[
\int_{\gamma} \langle V(x), \, dx \rangle = \int_{0}^{1} \langle V(\gamma(t)), \gamma'(t) \, dt \rangle.
\]

[[]] **Divergence**
Main article: Divergence

The divergence of a vector field on Euclidean space is a function (or scalar field). In three-dimensions, the divergence is defined by

$$\text{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

with the obvious generalization to arbitrary dimensions. The divergence at a point represents the degree to which a small volume around the point is a source or a sink for the vector flow, a result which is made precise by the divergence theorem.

The divergence can also be defined on a Riemannian manifold, that is, a manifold with a Riemannian metric that measures the length of vectors.

[] Curl

Main article: Curl (mathematics)

The curl is an operation which takes a vector field and produces another vector field. The curl is defined only in three-dimensions, but some properties of the curl can be captured in higher dimensions with the exterior derivative. In three-dimensions, it is defined by

$$\text{curl}\mathbf{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{e}_1 - \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \mathbf{e}_2 + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{e}_3.$$

The curl measures the density of the angular momentum of the vector flow at a point, that is, the amount to which the flow circulates around a fixed axis. This intuitive description is made precise by Stokes' theorem.

[] History

Magnetic field lines of an iron bar (magnetic dipole)

Vector fields arose originally in classical field theory in 19th century physics, specifically in magnetism. They were formalized by Michael Faraday, in his concept of lines of force, who
emphasized that the field itself should be an object of study, which it has become throughout physics in the form of field theory.

In addition to the magnetic field, other phenomena that were modeled as vector fields by Faraday include the electrical field and light field.

[] Flow curves

Main article: Integral curve

Consider the flow of a fluid, such as a gas, through a region of space. At any given time, any point of the fluid has a particular velocity associated with it; thus there is a vector field associated to any flow. The converse is also true: it is possible to associate a flow to a vector field having that vector field as its velocity.

Given a vector field $V$ defined on $S$, one defines curves $\gamma(t)$ on $S$ such that for each $t$ in an interval $I$

$$\gamma'(t) = V(\gamma(t)).$$

By the Picard–Lindelöf theorem, if $V$ is Lipschitz continuous there is a unique $C^1$-curve $\gamma_x$ for each point $x$ in $S$ so that

$$\gamma_x(0) = x,$$
$$\gamma'_x(t) = V(\gamma_x(t)) \quad (t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}).$$

The curves $\gamma_x$ are called flow curves of the vector field $V$ and partition $S$ into equivalence classes. It is not always possible to extend the interval $(-\varepsilon, +\varepsilon)$ to the whole real number line. The flow may for example reach the edge of $S$ in a finite time. In two or three dimensions one can visualize the vector field as giving rise to a flow on $S$. If we drop a particle into this flow at a point $p$ it will move along the curve $\gamma_p$ in the flow depending on the initial point $p$. If $p$ is a stationary point of $V$ then the particle will remain at $p$.

Typical applications are streamline in fluid, geodesic flow, and one-parameter subgroups and the exponential map in Lie groups.

[] Complete vector fields

A vector field is complete if its flow curves exist for all time. In particular, compactly supported vector fields on a manifold are complete. If $X$ is a complete vector field on $M$, then the one-parameter group of diffeomorphisms generated by the flow along $X$ exists for all time.

[] Difference between scalar and vector field
The difference between a scalar and vector field is not that a scalar is just one number while a vector is several numbers. The difference is in how their coordinates respond to coordinate transformations. A scalar is a coordinate whereas a vector can be described by coordinates, but it is not the collection of its coordinates.

[] Example 1

This example is about 2-dimensional Euclidean space \((\mathbb{R}^2)\) where we examine Euclidean \((x, y)\) and polar \((r, \theta)\) coordinates (which are undefined at the origin). Thus \(x = r \cos \theta\) and \(y = r \sin \theta\) and also \(r^2 = x^2 + y^2\), \(\cos \theta = x/(x^2 + y^2)^{1/2}\) and \(\sin \theta = y/(x^2 + y^2)^{1/2}\). Suppose we have a scalar field which is given by the constant function 1, and a vector field which attaches a vector in the \(r\)-direction with length 1 to each point. More precisely, they are given by the functions

\[
\begin{align*}
    s_{\text{polar}} : (r, \theta) &\mapsto 1, \\
    v_{\text{polar}} : (r, \theta) &\mapsto (1, \theta).
\end{align*}
\]

Let us convert these fields to Euclidean coordinates. The vector of length 1 in the \(r\)-direction has the \(x\) coordinate \(\cos \theta\) and the \(y\) coordinate \(\sin \theta\). Thus in Euclidean coordinates the same fields are described by the functions

\[
\begin{align*}
    s_{\text{Euclidean}} : (x, y) &\mapsto 1, \\
    v_{\text{Euclidean}} : (x, y) &\mapsto (\cos \theta, \sin \theta) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).
\end{align*}
\]

We see that while the scalar field remains the same, the vector field now looks different. The same holds even in the 1-dimensional case, as illustrated by the next example.

[] Example 2

Consider the 1-dimensional Euclidean space \(\mathbb{R}\) with its standard Euclidean coordinate \(x\). Suppose we have a scalar field and a vector field which are both given in the \(x\) coordinate by the constant function 1,

\[
\begin{align*}
    s_{\text{Euclidean}} : x &\mapsto 1, \\
    v_{\text{Euclidean}} : x &\mapsto 1.
\end{align*}
\]

Thus, we have a scalar field which has the value 1 everywhere and a vector field which attaches a vector in the \(x\)-direction with magnitude 1 unit of \(x\) to each point.

Now consider the coordinate \(\xi := 2x\). If \(x\) changes one unit then \(\xi\) changes 2 units. Thus this vector field has a magnitude of 2 in units of \(\xi\). Therefore, in the \(\xi\) coordinate the scalar field and the vector field are described by the functions

\[
\begin{align*}
    s_{\text{unusual}} : \xi &\mapsto 1, \\
    v_{\text{unusual}} : \xi &\mapsto 2
\end{align*}
\]

which are different.
Generalizations

Replacing vectors by $p$-vectors ($p$th exterior power of vectors) yields $p$-vector fields; taking the dual space and exterior powers yields differential $k$-forms, and combining these yields general tensor fields.

Algebraically, vector fields can be characterized as derivations of the algebra of smooth functions on the manifold, which leads to defining a vector field on a commutative algebra as a derivation on the algebra, which is developed in the theory of differential calculus over commutative algebras.